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ANALYSIS OF SQUARE RETICULAR PATTERN FOR HYPARS

By
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Abstract

The analogous continuum approach is formulated, providing procedures for the analysis and design of reticulated (or discrete member) structures. The analogous continuum constitutes in effect a hypothetical solid material whose properties are functions of those of the reticular pattern under consideration. Analyses based on the analogous continuum approach necessitate application of a valid continuum theory, and offer an alternative to solutions by discrete member computer methods.

A single layer reticular pattern, suitable for shallow hyperbolic paraboloid shells (slope of edge members ≤¼) bounded by a warped rectangle, is studied in detail. The basic module (elementary unit) of this pattern consists of a square and its diagonals inscribed symmetrically within a larger square. Mathematical formulations are given for: (a) properties of the analogous continuum corresponding to the above reticular pattern, and (b) forces and moments in the members of the reticular pattern.

KEY WORDS: *analogous, analysis, continuum,* design, *discrete member, hyperbolic paraboloid,* module, reticulated, single layer, shallow, shells, structures, *theory.*

Introduction

In the field of structural analysis and design of latticed or reticulated structures (plates and shells) two basic approaches have been used, namely, (a) that of discrete members, and (b) that of the "analogous continuum".

According to the first approach, the structure is conventionally analyzed as a plane or space framework consisting of the physical discrete members. Such analyses are almost invariably performed through computer facilities on the basis of equilibrium and compatibility conditions at the joints. An early example of this approach is that of analyzing a radar antenna (5). According to the second approach, the reticulated structure is transformed into a hypothetical material, called the "analogous continuum" and is treated on the basis of a valid continuum theory. Early examples of this treatment are those of del Pozo (1), Kloppel and Schardt (2) and Mitchell (4).

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The principles and potentialities of the analogous continuum approach have been recognized and outlined in previous work by Wright (6) (7), and they are formulated in the first part of this paper. In the second part of the paper, properties are established for a single layer reticular pattern whose basic module consists of a square and its diagonals inscribed symmetrically within a larger square. This module was found suitable for shallow hyperbolic paraboloid shells, bounded by a warped rectangle.

Part 1: General Considerations of the Analogous Continuum Approach

1. Definition of Basic Module

Basic modules (elementary units) of reticular patterns, are defined as the smallest geometric integral entities, successive repetition of which would generate the reticular pattern and ultimately would establish the geometric shape of the structure.

A reticulated structure may be simulated (by analogy to chemistry) by a thin sheet of solid material, formed through two-dimensional polymerization. The members of the reticular pattern, its nodes and its basic module would then be analogous respectively to the chemical bonds, the atoms, and the characteristic molecule of the polymerized material.

2. Outline of Method of Analysis

The method of analysis, in conformity with the analogous continuum approach, consists of the following stages:

1. A suitable and valid continuum theory is adopted for application.
2. A reticular pattern is adopted such that: (a) it suits the geometric configuration of the structure, (b) it consists of basic modules of configurations and boundaries compatible with the mathematics of the continuum theory adopted in stage 1.
3. A hypothetical solid material, termed the "analogous continuum", is established. Its physical thickness and elastic properties are determined on the basis of equality of deformations of the basic module and an equal size element of the analogous continuum, under identical forces.
4. The continuum theory adopted in stage 1 is applied to the structure of same geometry as the original one, but assumed to consist of the analogous continuum established in stage 3.
5. General stability of the structure may be assessed on the basis of valid buckling theories, and structure actions (stress resultants, deflections, displacements) may be determined according to the results of stage 4.
6. Stress resultants determined in stage 5 are suitably transformed into forces of the members of the reticular pattern.

3. Advantages and Shortcomings

The advantages of the analogous continuum approach as compared with the discrete member approach may be summarized as follows:

1. Stress fields become easily surveyable and regions of high and low stresses may be detected.
2. General stability conditions may be formulated and examined on the basis of valid buckling theories.
3. The factor of limiting computer capacity, which may arise due to a large number of simultaneous equations to be solved, is absent.
4. Several solutions, corresponding to reticular patterns of various configurations, may be obtained, with relatively little effort, for the purpose of optimization of conditions pertaining to material usage and fabrication.

The shortcomings of the analogous continuum approach are inherently identical with those of the continuum theory on which it is based, namely, possible inability to deal with: (a) partial, nonuniform or point loading, (b) geometric discontinuities, (c) specialized boundary and support conditions, (d) appropriate modification of the analogous continuum at over-stressed regions.

Part 2: A Reticular Pattern Suitable for Shallow Hypars

1. Description of the Reticular Pattern

A shallow hyperbolic paraboloid bounded by a warped rectangle is shown in Figure 1. A suitable reticular pattern for this shell form is shown in Figure 2; lines 1-1′, 2-2′, 3-3′ etc. and a-a′, b-b′, c-c′ etc. occupy the position of straight generators, while lines 2-b, 4-d etc. and 2-e′, 4-c′ etc. are tension and compression diagonals. A typical basic module ABCD of the reticular pattern is shown shaded (Figure 2); its configuration and orientation is compatible with continuum theories related to the OXYZ reference system. Induced angles between members of the pattern are either 90° or 45°.

This reticular pattern possesses the following advantages: (a) it is symmetric with respect to the shell diagonals and shell generators and therefore it appears visually harmonious (when exposed), (b) it provides load points at the intersection of tension and compression diagonals so that superimposed loads can be taken by arch action, rather than by beam action, (c) there are not intervening superimposed loads within the basic module which would necessitate additional computations in evaluating forces in the members of the pattern, (d) it provides both a basic module which may be assumed planar, and an analogous continuum which is isotropic, as shown in further sections of the paper.

2. Properties of the Analogous Continuum

A. Assumptions of Analysis.

1. All straight generators 1-1′, 2-2′ etc. and a-a′, b-b′ etc. (Figure 2) have moments of inertia and cross-sectional areas designated by I_1 and A_1 respectively. All tension and compression diagonals 2-b, 4-d etc. and 2-e′, 4-c′ etc. (Figure 2) have moments of inertia and cross-sectional areas designated by I_2 and A_2 respectively. Moments of inertia are those effective in bending in a direction normal to the shell surface.
2. All members of the pattern have the same modulus of elasticity E.
3. All joints of the reticular pattern (Figure 2) are hinged about pins perpendicular to the shell surface but rigid in all other directions. A consequence of this assumption is that extensional and in plane angular deformations produced respectively by direct forces and in plane shears are resisted by extensional rigidities of the members.
4. Extensional and bending stiffnesses, of the members of the pattern, are not altered by intervening joints.
5. Transverse shear deformations of the analogous continuum or the members of the reticular pattern are negligible.
6. The basic module, represented typically by ABCD (Figure 2), is a plane element. In fact, the perimetric sides of the basic module form a square of such a small warpage that the tangential bending and twisting moment vectors, direct force and, in plane shear stress resultants may be safely considered as lying on a common plane. For instance, for the case of a shell bounded by a warped square whose sides form a slope of 0.25 with the OXY plane (Figure 1), containing ten basic modules in each direction, the angle between opposite sides of the basic module (AB-DC, AD-BC, Figure 2) is less than 3°.

B. Constants of the Analogous Continuum

The basic module is shown in detail in Figure 3. The halved values of moment of inertia and cross-sectional area of AB, BC, CD, DA reflect the fact that these members are shared by abutting basic modules. On the basis of the analysis shown in Appendix II-Derivations of Relationships, taking into account extensional, bending, and in plane shearing deformations, the following results have been obtained:

$$E'_x = E'_y = \frac{4\sqrt{2}\,\alpha^2 + 4\,\alpha}{2\sqrt{3}\sqrt{(2\sqrt{2}\,\alpha + 1)(2\sqrt{2}\,\iota + 1)}} \cdot \frac{A_2\sqrt{A_2}\ \ E}{L\sqrt{I_2}} \quad \cdot \qquad 1$$

$$h' = 2\sqrt{3\,\frac{2\sqrt{2}\,\iota + 1}{2\sqrt{2}\,\alpha + 1}\cdot\frac{I_2}{A_2}} \quad . \quad . \quad . \quad 2$$

$$\nu' = \frac{1}{2\sqrt{2}\,\alpha + 1} \quad . \quad . \quad . \quad 3$$

$$G'_x = G'_y = \frac{\sqrt{2}\,A_2\,E}{L\,h'} \quad . \quad . \quad . \quad 4$$

in which E, h, ν, G, L are the modulus of elasticity, thickness, Poisson's ratio, shear modulus, and side length of the square basic module respectively; the prime ($'$) refers to the analogous continuum; subscripts x, y designate association with directions OX, OY respectively; α and ι are ratios given by

$$\alpha = \frac{A_1}{A_2} \quad . \quad . \quad . \quad . \quad 5$$

$$\iota = \frac{I_1}{I_2} \quad . \quad . \quad . \quad 6$$

Equations 1 and 4 indicate that the analogous continuum is isotropic.

For the special case for which all members of the pattern have the same properties, i.e. $\alpha = \iota = 1$ the above expressions become

$$E'_x = E'_y = 0.73\,\frac{A\sqrt{A}\,E}{L\sqrt{I}} \quad . \quad . \quad . \quad 7$$

$$h' = 2\sqrt{3\,\frac{I}{A}} \quad . \quad . \quad . \quad 8$$

$$\nu' = 0.26 \quad . \quad . \quad . \qquad 9$$

$$G'_x = G'_y = \frac{A\sqrt{A}\ E}{L\sqrt{6I}} \quad . \quad . \qquad 10$$

in which A, I are the common cross sectional area and moment of inertia respectively, of the members of the pattern.

3. Forces and Moments in the Members of the Pattern

Results determined in Appendix II-Derivations of Relationships, are summarized below. Positive numerical results, obtained after substitution into the following expressions, indicate that forces are tensile and moments are sagging. Stress resultants for substitution into the following expressions are positive when they are of the sense shown in Figure 4.

A. Axial Forces

Member designation is that shown in Figure 3.

Member EG:

$$P = \frac{L}{4\sqrt{2}\,\alpha + 4}\left\{(2\sqrt{2}\,\alpha + 1)N_y - N_x\right\} \quad . \quad . \quad . \qquad 11$$

Member HF:

$$P = \frac{L}{4\sqrt{2}\,\alpha + 4}\left\{(2\sqrt{2}\,\alpha + 1)N_x - N_y\right\} \quad . \quad . \quad . \qquad 12$$

Members EH, FG:

$$P = \frac{\sqrt{2}\,L}{4\sqrt{2}\,\alpha + 4}\left\{N_x + N_y\right\} + \frac{LN_{xy}}{\sqrt{2}} \quad . \quad . \quad . \qquad 13$$

Members EF, HG:

$$P = \frac{\sqrt{2}\,L}{4\sqrt{2}\,\alpha + 4}\left\{N_x + N_y\right\} - \frac{LN_{xy}}{\sqrt{2}} \quad . \quad . \quad . \qquad 14$$

in which P are axial forces in the members of the reticular pattern; N_x, N_y are direct forces, and N_{xy} are "in plane shear" forces, per unit length of the analogous continuum (Figure 4).

B. Bending Moments

Member EG:

$$M = \frac{\sqrt{2}\,\iota}{2\sqrt{2}\,\iota + 1} L M_y \quad . \quad . \quad . \qquad 15$$

Member HF·

$$M = \frac{\sqrt{2}\,\iota}{2\sqrt{2}\,\iota + 1} L M_x \quad . \quad . \quad . \qquad 16$$

Members EH, FG:

$$M = \frac{L\sqrt{2}}{4\sqrt{2}\,\iota + 2}\{M_x + M_y\} + \frac{L M_{xy}}{\sqrt{2}} \quad . \qquad 17$$

Members EF, HG:

$$M = \frac{L\sqrt{2}}{4\sqrt{2}\,\iota + 2}\{M_x + M_y\} + \frac{L M_{xy}}{\sqrt{2}} \quad . \qquad 18$$

in which M are moments in the members of the reticular pattern; M_x, M_y are bending moments, and M_{xy} are twisting moments per unit length of the analogous continuum (Figure 4).

C. Transverse Shears

Member EG:

$$Q = \frac{L\sqrt{2}\,\iota}{2\sqrt{2}\,\iota + 1} V_y \quad . \quad . \quad . \qquad 19$$

Member HF:

$$Q = \frac{L\sqrt{2}\,\iota}{2\sqrt{2}\,\iota + 1} V_x \quad . \quad . \quad . \quad 20$$

Members EH, EF, GH, GF:

$$Q = \frac{L}{2(2\sqrt{2}\,\iota + 1)} (V_x + V_y) \quad . \quad . \quad . \quad 21$$

in which Q are transverse shear forces in the members of the reticular pattern; V_x, V_y are transverse shear forces per unit length of the analogous continuum (Figure 4).

Summary and Conclusions

The "analogous continuum" approach constitutes an alternative to solutions of reticulated (discrete member) structures by computer methods. Formulation of this approach, given in this paper, consists principally of: (a) replacing the proposed reticular pattern by an equivalent hypothetical material termed the "analogous continuum", (b) applying a valid continuum theory on a structure consisting of the analogous continuum, (c) determining stress resultants and transforming them into forces and moments in the members of the reticular pattern.

Principal advantages of this approach, as compared to discrete member computer solutions, are: (a) surveyability of stress fields and overall stability conditions, and (b) absence of the condition of limiting computer capacity, which may possibly arise due to the large number of simultaneous equations to be solved.

A reticular pattern, suitable for shallow hyperbolic paraboloid shells bounded by a warped rectangle, is studied in detail. The basic module of this pattern consists of a square and its diagonals inscribed symmetrically within a larger square. Properties of the analogous continuum corresponding to the above reticular pattern, and forces, moments and shears in the members of the pattern, are given by mathematical formulations.

Appendix I - References

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Appendix II - Derivations of Relationships

DETERMINATION OF PROPERTIES OF ANALOGOUS CONTINUUM

1. Direct Force Actions

Extensional characteristics are determined by use of the theorem of minimum total potential (3). The potential energy of external forces LN_y acting on the basic module (Figure A.1) is

$$V = -\delta_y \, L \, N_y \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{A.1}$$

in which V is the potential energy of external forces; δ_y is the deformation in the OY direction*. The strain energy stored in the members of the module is given by the following expressions:

Member AHD:

$$U_1 = \frac{1}{2}\,\frac{\delta_y^2\left(\frac{A_1}{2}\right)E}{L} = \frac{\delta_y^2 \, A_1 E}{4L} \quad \ldots\ldots\ldots \quad \text{A.2}$$

Member AEB:

$$U_2 = \frac{1}{2}\,\frac{\delta_x^2\left(\frac{A_1}{2}\right)E}{L} = \frac{\delta_x^2 A_1 E}{4L} \quad \ldots\ldots\ldots \quad \text{A.3}$$

Member EH:

$$U_3 = \frac{1}{2}\left(\frac{\delta_y}{2\sqrt{2}} + \frac{\delta_x}{2\sqrt{2}}\right)^2 \cdot \frac{A_2 E}{L/\sqrt{2}} = \frac{\sqrt{2}}{16}(\delta_y + \delta_x)^2 \frac{A_2 E}{L} \quad \ldots \quad \text{A.4}$$

*Identification of symbols already shown in the main body of the paper is not repeated in this Appendix.

Member EG:

$$U_4 = \frac{1}{2} \cdot \frac{\delta_y^2 A_1 E}{L} = \frac{\delta_y^2 A_1 E}{2L} \quad \ldots\ldots\ldots\ldots \quad \text{A.5}$$

Member HF:

$$U_5 = \frac{1}{2} \cdot \frac{\delta_x^2 A_1 E}{L} = \frac{\delta_x^2 A_1 E}{2L} \quad \ldots\ldots\ldots\ldots \quad \text{A.6}$$

in which the U terms designate strain energy stored in the corresponding member; δ_x is the deformation in the OX direction.

The total strain energy stored in the basic module with terms pertaining to member types, two of each of AHD and AEB, four of EH, and one of each of EG and HF is

$$U_T = 2U_1 + 2U_2 + 4U_3 + U_4 + U_5 \quad \ldots\ldots \quad \text{A.7}$$

in which U_T is the total strain energy stored in the basic module. Substituting the values of strain energy terms of Eqs. A.2 through A.6 into Eq. A.7 and combining the expression thus obtained with Eq. A.1, the total potential $V + U_T$ is given by

$$V+U_T = -\delta_y L N_y + \frac{\delta_y^2 A_1 E}{2L} + \frac{\delta_x^2 A_1 E}{2L} + \frac{\sqrt{2}}{4}(\delta_y + \delta_x)^2 \frac{A_2 E}{L} + \frac{\delta_y^2 A_1 E}{2L} + \frac{\delta_x^2 A_1 E}{2L} \quad \ldots \quad \text{A.8}$$

For the conditions $\partial(V + U_T)/\partial\delta_x = 0$, and $\partial(V + U_T)/\partial\delta_y = 0$, Eq. A.8 provides a system of two equations, solution of which gives

$$\delta_x = -\frac{1}{4\sqrt{2}\alpha^2 + 4\alpha} \cdot \frac{L^2 N_y}{A_2 E} \quad \ldots\ldots\ldots\ldots \quad \text{A.9}$$

$$\delta_y = \frac{2\sqrt{2}\,\alpha + 1}{4\sqrt{2}\alpha^2 + 4\alpha} \cdot \frac{L^2 N_y}{A_2 E} \quad \ldots\ldots\ldots\ldots \quad \text{A.10}$$

From Eqs. A.9 and A.10 Poisson's ratio is determined as

$$\nu' = \frac{\delta_x}{\delta_y} = \frac{1}{2\sqrt{2}\alpha + 1} \quad \ldots\ldots\ldots\ldots \quad \text{A.11}$$

Hooke's law in the OY direction for the analogous continuum is

$$\varepsilon_y = \frac{\sigma_y}{E'_y} - \nu'_y \frac{\sigma_x}{E'_x} \quad \text{A.12}$$

in which ε is the strain; σ are stresses. Also

$$\delta_y = \varepsilon_y L \quad \text{A.13}$$

$$\sigma_y = \frac{N_y}{h'} \quad \text{A.14}$$

Eliminating between Eqs. A.10 and A.13, and then eliminating N_y between the equation thus obtained and Eq. A.14

$$\sigma_y = \frac{\varepsilon_y}{h'} \cdot \frac{4\sqrt{2}\,\alpha^2+4\alpha}{2\sqrt{2}\,\alpha+1} \cdot \frac{A_2 E}{L} \quad \text{A.15}$$

Eliminating σ_y between Eqs. A.15 and A.12 for $\sigma_x = 0$

$$E'_y = \frac{4\sqrt{2}\,\alpha^2+4\alpha}{2\sqrt{2}\,\alpha+1} \cdot \frac{A_2 E}{Lh'} \quad \text{A.16}$$

By a similar process

$$E'_x = \frac{4\sqrt{2}\,\alpha^2+4\alpha}{2\sqrt{2}\,\alpha+1} \cdot \frac{A_2 E}{Lh'} \quad \text{A.17}$$

2. Bending Moment Actions

Bending characteristics are determined by use of moment-curvature relationships. The minimum radius of curvature R_y of the basic module, under the action of bending moments LM_y (Figure A.2), is, naturally, in the direction of members AHD, EG, BFC. The radius of curvature R_c, pertaining to members EH and EF is then,

$$R_c = \frac{R_y}{\cos^2 45^o} = 2R_y \quad \text{A.18}$$

On the basis of the above considerations

$$M_1 = \frac{E(\frac{I_1}{2})}{R_y} = \frac{EI_1}{2R_y} \quad \text{A.19}$$

$$M_2 = \frac{EI_1}{R_y} \quad \ldots\ldots \qquad \text{A.20}$$

$$M_3 = \frac{EI_2}{R_c} = \frac{EI_2}{2R_y} \quad \ldots\ldots \qquad \text{A.21}$$

in which M_1, M_2, M_3 are the magnitudes of the moment vectors which are normal to members AHD, EG, EH respectively. The component of moment M_3 normal to the OY axis has a magnitude M'_3 given by

$$M_3' = \frac{EI_2}{2R_y}\cos 45^\circ = \frac{EI_2}{2\sqrt{2}\ R_y} \quad \ldots\ldots \qquad \text{A.22}$$

The total moment M_T acting on the face of the module is

$$M_T = 2M_1 + M_2 + 2M_3' \quad \ldots\ldots \qquad \text{A.23}$$

For equilibrium

$$M_T = LM_y \quad \ldots\ldots \qquad \text{A.24}$$

Combining Eqs. A.19, A.20, A.22, A.23, A.24 and setting $I_1/I_2 = \iota$, gives

$$LM_y = \frac{2\sqrt{2}\,\iota + 1}{\sqrt{2}} \cdot \frac{EI_2}{R_y} \quad \ldots\ldots \qquad \text{A.25}$$

Considering a square element of the analogous continuum of same radius of curvature R_y and size L x L, as the basic module, the bending moment-flexural rigidity relationship is given by

$$LM_y = \frac{L}{R_y} \cdot \frac{E_y' h'^3}{12(1 - \nu'^2)} \quad \ldots\ldots \qquad \text{A.26}$$

Substituting in Eq. A.26 the values of E'_y and ν' from Eqs. A.17 and A.11 respectively, eliminating the term LM_y between the equation thus obtained and Eq. A.25 and solving for h'

$$h' = 2\sqrt{3\,\frac{(2\sqrt{2}\,\iota + 1)}{(2\sqrt{2}\,\alpha + 1)} \cdot \frac{I_2}{A_2}} \quad \ldots\ldots \qquad \text{A.27}$$

Substituting the value of h′ from Eq. A.27 into Eq. A.16 or Eq. A.17

$$E_x^{\prime} = E_y^{\prime} = \frac{4\sqrt{2}\,\alpha^2 + 4\,\alpha}{2\sqrt{3}\sqrt{(2\sqrt{2}\,\alpha + 1)(2\sqrt{2}\,\iota + 1)}} \cdot \frac{A_2\sqrt{A_2}\,E}{L\sqrt{I_2}} \qquad \text{A.28}$$

3. "In Plane Shear" Actions

In accordance with assumption 3 of subsection "A. Assumptions of Analysis", shear forces are resisted by the members of the sides of the inscribed square only (Figure A.3). Therefore

$$P_1 = \frac{LN_{xy}}{\sqrt{2}} \quad \cdots\cdots\cdots \qquad \text{A.29}$$

$$P_2 = -\frac{LN_{xy}}{\sqrt{2}} \quad \cdots\cdots\cdots \qquad \text{A.30}$$

in which P_1, P_2 are axial forces in members EH, EF respectively, and the minus sign indicates compression. Corresponding half deformations of the above members, are given by

$$bc = ac = \frac{L^2 N_{xy}}{4\,A_2 E} \quad \cdots\cdots\cdots \qquad \text{A.31}$$

For the small right angle triangle acb (Figure A.3)

$$ab = \sqrt{2}\,bc = \frac{L^2 N_{xy}}{2\sqrt{2}\,A_2 E} \quad \cdots\cdots \qquad \text{A.32}$$

also

$$\gamma_{xy} = \frac{ab}{L/2} \quad \cdots\cdots\cdots \qquad \text{A.33}$$

in which γ_{xy} is the angular deformation of the basic module. Combining Eqs. A.32 and A.33

$$\gamma_{xy} = \frac{L^2 N_{xy}}{2\sqrt{2}\,A_2 (L/2)\,E} = \frac{LN_{xy}}{\sqrt{2}\,A_2 E} \quad \cdots\cdots \qquad \text{A.34}$$

Considering that an element of the analogous continuum of same size as the basic module L x L, will deform by the same amount γ_{xy}

$$\gamma_{xy} = \frac{LN_{xy}/ Lh'}{G'} = \frac{N_{xy}}{h' G'} \quad \cdots \quad A.35$$

Eliminating γ_{xy} between Eqs. A.34 and A.35

$$G' = \frac{\sqrt{2}\, A_2 E}{Lh'} \quad \cdots \quad A.36$$

If the analogous continuum is considered as an ordinary engineering material, then

$$G' = \frac{E'}{2(1 + \nu')} \quad \cdots \quad A.37$$

Through combination of Eqs. A.11, A.16, A.27, A.36 with A.37, it is deduced that, Eq. A.37 is fulfilled only if $\alpha = 2$.

DETERMINATION OF FORCES AND MOMENTS IN THE MEMBERS OF THE RETICULAR PATTERN

1. General

Positive signs indicate that forces are tensile and moments are sagging. Stress resultants for substitution into the following expressions are positive when they are of the sense shown in Figure 4. Member designation is that shown in Figure 3.

2. Axial Forces due to N_x and N_y

Member EG:
Considering the basic module acted on by forces LN_y (Figure A.1), the force-deformation relationship is

$$P = \frac{\delta_y}{L} EA_1 \quad \cdots \quad A.38$$

Combining Eqs. A.10 and A.38

$$P = \frac{2\sqrt{2}\,\alpha + 1}{4\sqrt{2}\,\alpha + 4} LN_y \quad \cdots \quad A.39$$

Considering the basic module acted on by forces LN_x and combining Eq. A. 38, which still holds, with Eq. A.9 holding rotationally for δ_y and N_x (instead of δ_x and N_y)

$$P = -\frac{1}{4\sqrt{2}\,\alpha + 4} LN_x \quad \cdots \quad A.40$$

Member HF:
By similar considerations as for member EG, expressions for axial forces are obtained, given by

$$P = \frac{2\sqrt{2}\,\alpha + 1}{4\sqrt{2}\,\alpha + 4} LN_x \quad \ldots\ldots\ldots \quad A.41$$

$$P = -\frac{1}{4\sqrt{2}\,\alpha + 4} LN_y \quad \ldots\ldots\ldots \quad A.42$$

Members HE, HG, FE, FG:
Considering the basic module acted on by forces LN_y (Figure A.1), the deformation Δ is given by

$$\Delta = \frac{\delta_x}{2\sqrt{2}} + \frac{\delta_y}{2\sqrt{2}} \quad \ldots\ldots\ldots \quad A.43$$

Combining Eqs. A.43, A.9, A.10

$$\Delta = \frac{1}{4\sqrt{2}\,\alpha + 4} \cdot \frac{L^2 N_y}{A_2 E} \quad \ldots\ldots\ldots \quad A.44$$

The force-deformation relationship for the above members, is given by

$$P = \frac{\Delta}{L/\sqrt{2}} EA_2 \quad \ldots\ldots\ldots \quad A.45$$

Combining Eqs. A.44 and A.45

$$P = \frac{\sqrt{2}}{4\sqrt{2}\,\alpha + 4} LN_x \quad \ldots\ldots\ldots \quad A.46$$

Similarly, considering the basic module acted on by forces LN_y

$$P = \frac{\sqrt{2}}{4\sqrt{2}\,\alpha + 4} LN_y \quad \ldots\ldots\ldots \quad A.47$$

3. Axial Forces due to N_{xy}

Expressions for these axial forces are given by Eq. A.29 for members EH, FG, and by Eq. A.30 for members EF, HG.

4. Bending Moments due to M_x and M_y

Member EG:
Considering the basic module acted on by bending moments LM_y and combining the relevant Eqs. A.20 and A.25

$$M = \frac{\sqrt{2}\,\iota}{2\sqrt{2}\,\iota + 1} LM_y \quad \cdots\cdots\cdots \quad \text{A.48}$$

Member HF:
For the case in which the basic module is acted on by bending moments LM_x, an expression for the moment in member HF may be obtained, analogous to Eq. A.48, containing M_x instead of M_y.

Members EH, FG, EF, HG:
Considering the basic module acted on by bending moments LM_y and combining the relevant Eqs. A.21 and A.25

$$M = \frac{\sqrt{2}}{4\sqrt{2}\,\iota + 2} LM_y \quad \cdots\cdots\cdots \quad \text{A.49}$$

An expression analogous to Eq. A.49, containing M_x instead of M_y, may be obtained for the bending moment in the above members for the case in which the basic module is acted on by bending moments LM_x.

5. Bending Moments due to M_{xy}

In conformity with the assumption that the torsional rigidity of the members of the module is negligible, twisting moments LM_{xy} are resisted by the bending stiffness of the members HE, HG, FE, FG (Figure 3). Thus, the bending moment in all of the above members is given by

$$M = \frac{LM_{xy}}{\sqrt{2}} \quad \cdots\cdots \quad \text{A.50}$$

6. Transverse Shear Forces due to V_x, V_y

The basic module is shown in Figure A.4, under the action of transverse shears LV_y. Use is made of Eq. A-51

$$Q = \frac{dM}{d\ell} \quad \cdots\cdots \quad \text{A.51}$$

in which Q is the shear force in the member; ℓ is the coordinate in the direction of its length. Considering member AHD (for which ℓ of Eq. A.51 is equal to y) and combining Eqs. A.51 and A.19

$$Q_1 = \frac{EI_1}{2} \cdot \frac{d}{dy}\left(\frac{1}{R_y}\right) \quad . \quad . \quad . \quad . \qquad A.52$$

Substituting into Eq. A.51 K for $\frac{d}{dy} \cdot \left(\frac{1}{R_y}\right)$

$$Q_1 = \frac{EI_1}{2} K \quad . \quad . \quad . \qquad A.53$$

By a similar process, for member EG,

$$Q_2 = EI_1 K \quad . \quad . \quad . \quad . \qquad A.54$$

For member EH,

$$\ell = y' = \frac{y}{\cos 45^{o}} = y\sqrt{2} \quad . \quad . \qquad A.55$$

Combining Eqs. A.21, A.51, A.55 and substituting K for $\frac{d}{dy}\left(\frac{1}{R_y}\right)$

$$Q_3 = \frac{EI_2}{2\sqrt{2}} K \quad . \quad . \quad . \quad . \quad . \qquad A.56$$

For equilibrium

$$2Q_1 + Q_2 + 2Q_3 = LV_y \quad . \quad . \quad . \quad . \quad . \qquad A.57$$

Substituting in Eq. A.57 the values of Q_1, Q_2, Q_3 from Eqs. A.53, A.54, A.56 respectively, solving for K, substituting the value of K thus determined back into Eqs. A.53, A.54, A.56, the following expressions are obtained

$$Q_1 = \frac{\sqrt{2}\,\iota}{2(2\sqrt{2} + 1)} LV_y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad A.58$$

$$Q_2 = \frac{\sqrt{2}\,\iota}{2\sqrt{2}\,\iota + 1} LV_y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad A.59$$

$$Q_3 = \frac{1}{2(2\sqrt{2}\,\iota + 1)} LV_y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad A.60$$

Analogous expressions may be obtained for the case in which the basic module is acted on by transverse shears LV_x.

Appendix III - Notation

The following symbols have been adopted for use in this paper:

A = cross-sectional area of members of reticular pattern; for significance of subscripts see Figure 3;
E = modulus of elasticity;
G = shear modulus;
I = moment of inertia of members of reticular pattern; for significance of subscripts see Figure 3;
K = $\frac{d}{dy}\left(\frac{1}{R_y}\right)$;
L = side length of square basic module;
M = bending moments in members of reticular pattern;
M_x, M_y = bending moments per unit length of the analogous continuum, tangential to shell surface and normal to OX and OY axis respectively (Figure 4);
M_{xy} = twisting moment per unit length of the analogous continuum, tangential to shell surface (Figure 4);
N_x, N_y = direct forces per unit length of the analogous continuum, tangential to shell surface and parallel to OX and OY axis respectively, (Figure 4);
N_{xy}, N_{yx} = "in plane shear" forces per unit length of the analogous continuum, tangential to shell surface (Figure 4);
OX, OY, OZ, = axes of a right handed cartesian coordinate system OXYZ, (Figure 1);
P = axial forces in members of reticular pattern;
Q = transverse shear forces in members of reticular pattern, normal to shell surface;
R = radius of curvature;
U = strain energy;
V = transverse shear forces per unit length of the analogous continuum normal to shell surface (Figure 4); potential of external forces acting on an isolated basic module;
a, b = horizontal projections of half spans of shell on the OXY plane, (Figure 1);
c = rise of shell (Figure 1);
h = thickness;
k = shell shape parameter (k = z/xy);
ℓ = length variable in the direction of a member of the reticular pattern;
x, y, z = cartesian coordinates;

GREEK SYMBOLS

α = ratio of cross-sectional areas of members of reticular pattern, $= A_1/A_2$ (Figure 3);
γ = angular deformation of basic module acted on by "in plane shears";
δ = extensional deformation;
ε = strain at a point;
ι = ratio of moments of inertia of members of reticular pattern, $= I_1/I_2$ (Figure 3);
ν = Poisson's ratio;
σ = stress at a point;

INDICES

Superscripts

Primes (′) = denote properties of the analogous continuum;

Subscripts

T = signifies inclusion of all terms into the respective quantity;
x, y = signify association of the quantity with the OX and OY axis respectively;
xy = signify association of the quantity with "in plane shear" forces.

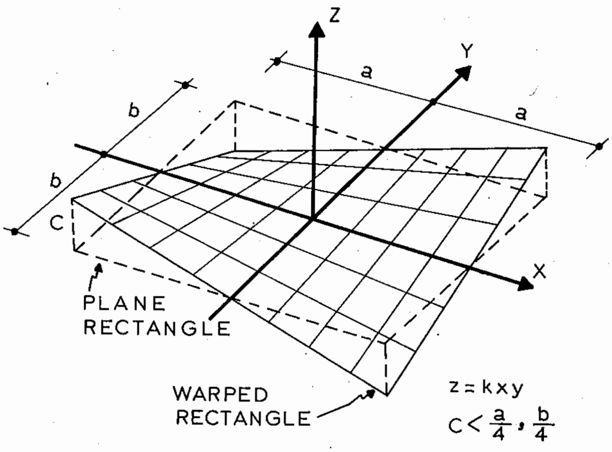

Figure 1: Isometric of Shallow Hypar Shell Bounded by Straight Generators

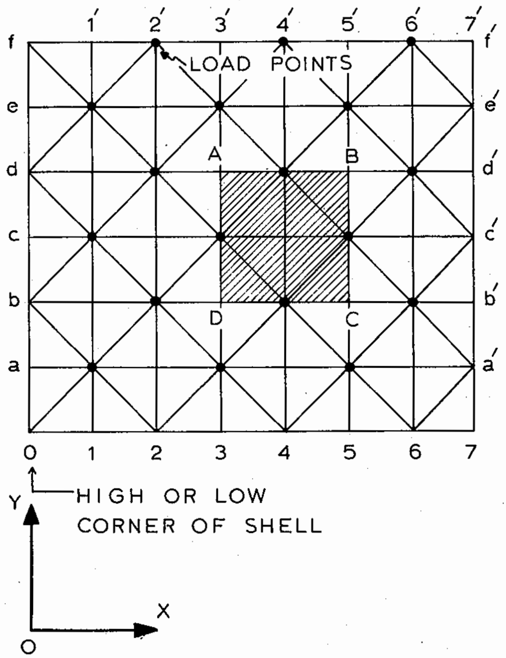

Figure 2: Reticular Pattern for Shallow Hypar Shell

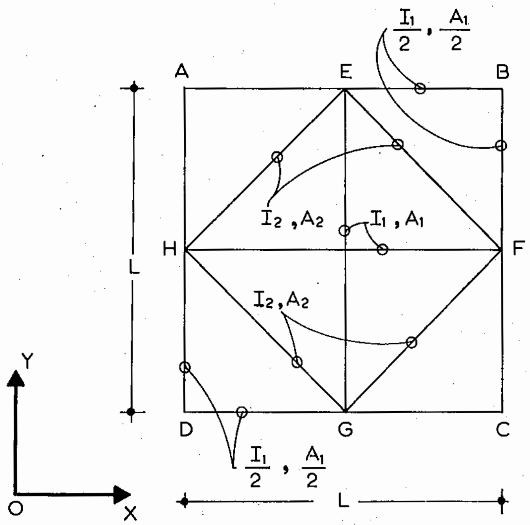

Figure 3: Basic Module Used in Analysis

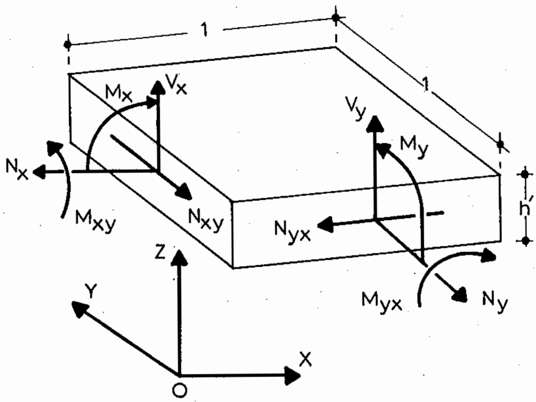

Figure 4: Positive Directions of Stress Resultants in an Element of Analogous Continuum

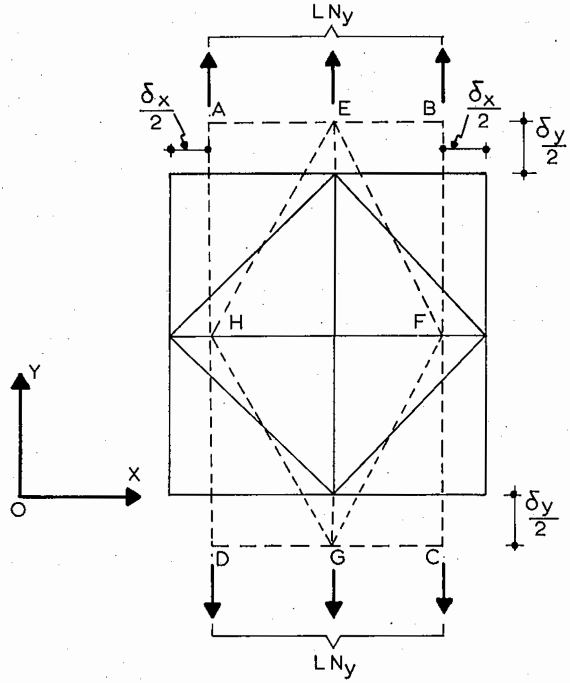

Figure A.1: Basic Module Acted on by Direct Forces

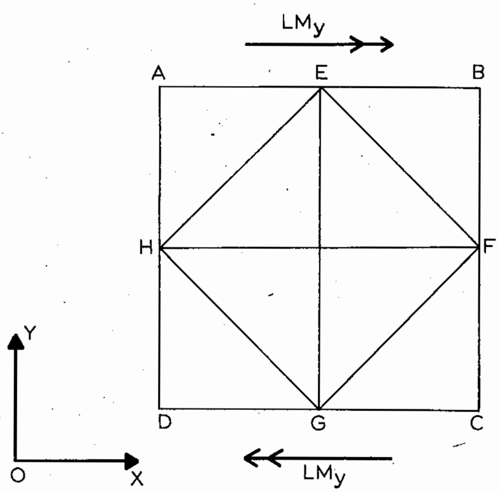

Figure A.2: Basic Module Acted on by Bending Moments

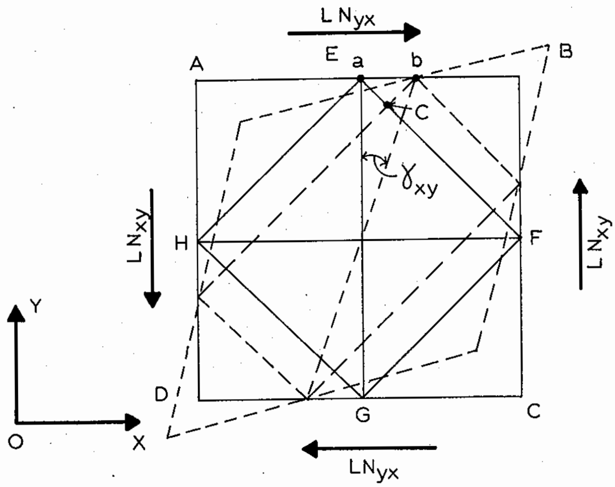

Figure A.3: Basic Module Acted on by "In Plane Shear" Forces

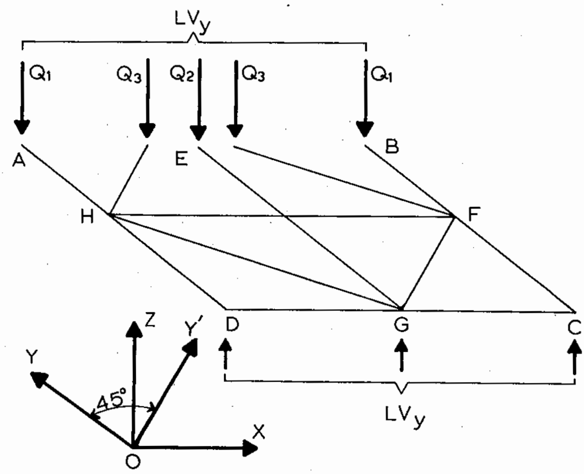

Figure A.4: Basic Module Acted on by Transverse Shear Forces