The Use of Back Analysis to Reduce Slope Failure Risk

Back analysis can be used to develop highly reliable computational models of slopes to determine failure risk and provide a means to develop reliable designs.

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According to the *Random House Dictionary, back analysis* is a form of analysis that "starts from a known result and works backward to prove a proposition."[1] Back analysis is frequently used in landslide repair. In that case, the known result is the fact that the landslide has occurred. The proposition to be proved involves the conditions at the time of failure.

In the best use of back analysis, the conditions at the time of failure — including slope geometry, external loads from water or other sources, soil weights, soil strengths, distribution of soils within the slope and piezometric levels within the slope — are estimated using all the information that can be assembled. For the conditions so established, the factor of safety with respect to stability of the slope is computed. If the computed value of the factor of safety, F, is equal to 1.0, the estimated conditions have been proven to be consistent with the fact that the slope failed. If the computed value of the factor of safety is not 1.0, the conditions are adjusted, still maintaining consistency with the available information. Through repeated trials a set of conditions is established that results in a factor of safety equal to 1.0 and is consistent with the known facts.

Back analysis is sometimes described as a process of calculating shear strength from a failure. In fact, it is both more and less than a calculation of shear strength. Back analysis is *more* than a simple calculation of shear strength because, in the end, a complete computational model has been developed that is consistent with the conditions at the time of the slope failure. The model includes geometry, loads, soil weights, soil strengths, piezometric levels and the method used to compute the factor of safety. While estimates and assumptions are always necessary to fill in gaps in knowledge, adjusting the model so that the factor of safety is equal to 1.0 for the failure condition produces a complete and reliable computational model.

Back analysis is *less* than a simple calculation of shear strength because, to compute the shear

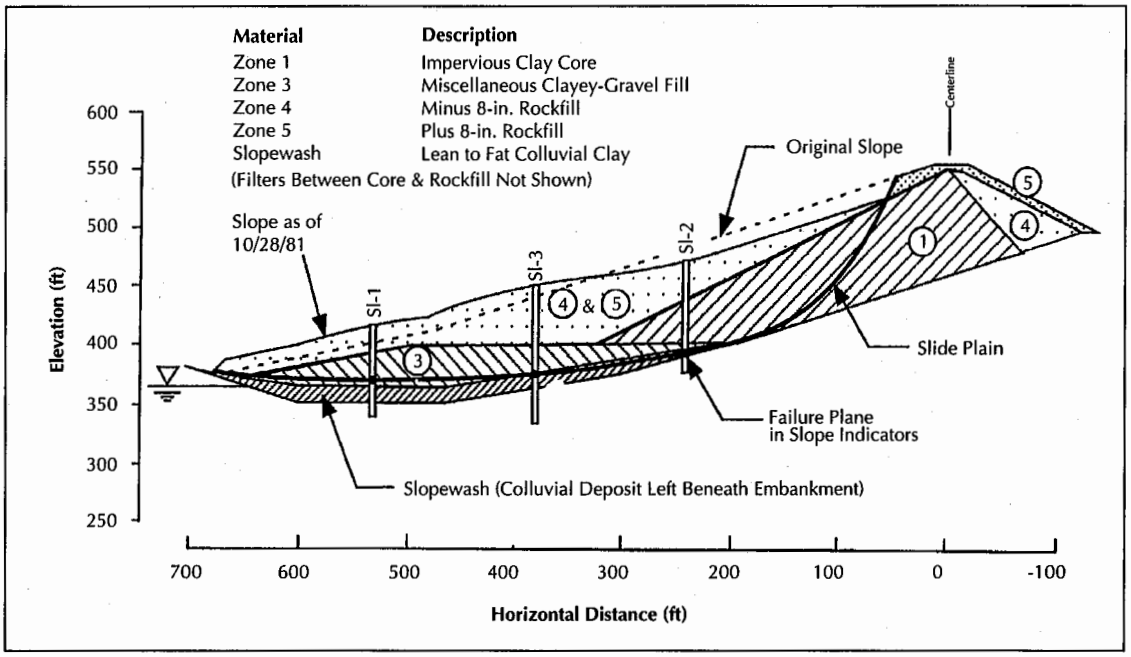

FIGURE 1. Slide in the upstream slope of the San Luis Dam.

strength of a particular soil zone within the slope, values of all of the other quantities must be estimated. If the estimates were changed, the "calculated" shear strength would also change to maintain the the end result of a factor of safety equal to 1.0.

Even fairly wide variations in some of the assumptions have little effect on the results of forward analyses used to design slope stabilization measures, provided that the back analysis results are used consistently in the forward analysis. In addition, all of the elements of the computational model should be reasonable and consistent with all known facts about the conditions at the time of failure.

Practical use of back analysis is best illustrated by showing how it has been applied. Case histories on three projects — San Luis Dam repair, Olmstead landslide stabilization and La Esperanza abutment drainage — provide an excellent means of describing how back analysis can be properly used.

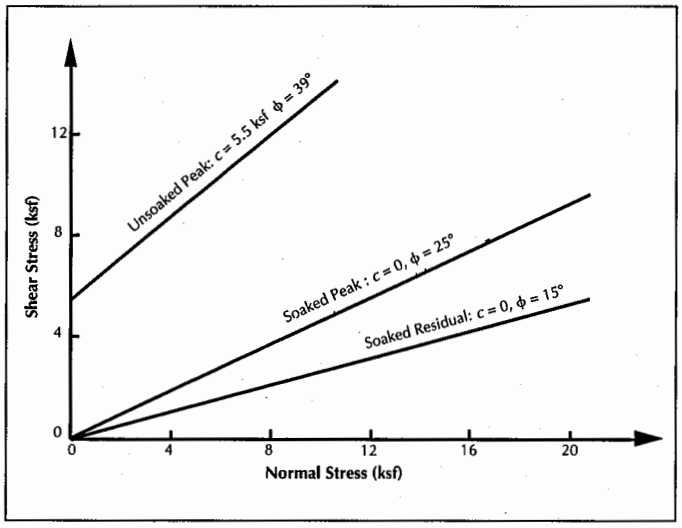

FIGURE 2. The strength of highly plastic clay slopewash from the foundation of the San Luis Dam.

San Luis Dam Repair

San Luis Dam impounds a pump-storage reservoir on the California Aqueduct System, about 90 miles southeast of San Francisco. In September 1981 a slide occurred in the upstream slope of the embankment as the reservoir was being drawn down.[2-4] A cross-section through the embankment where the slide occurred is shown in Figure 1.

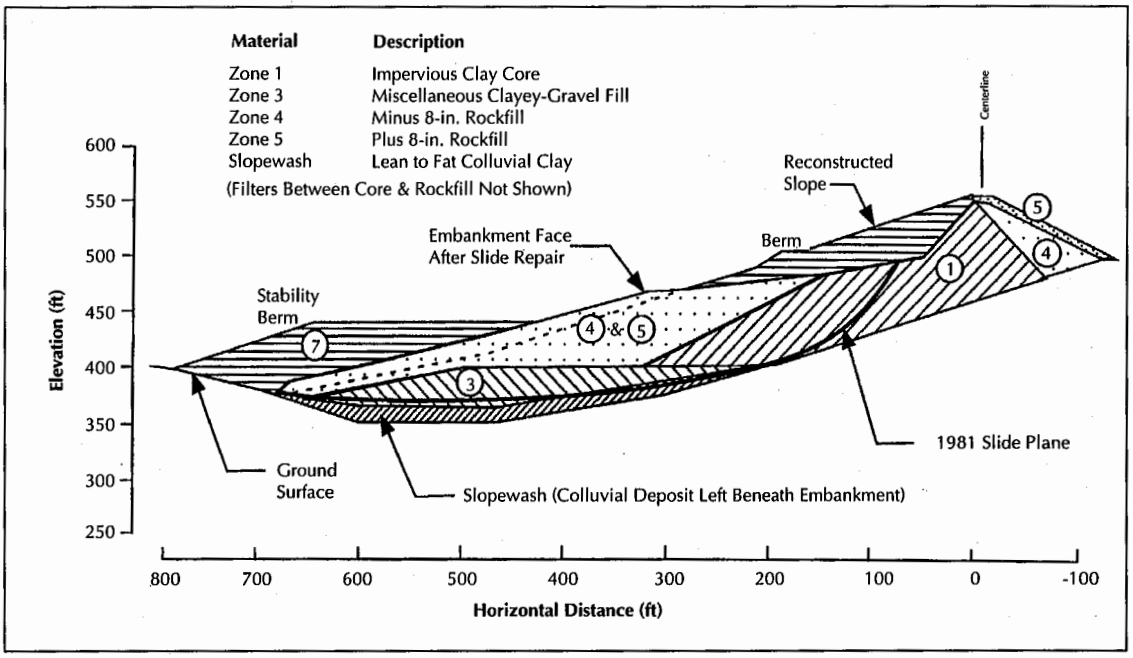

FIGURE 3. The San Luis Dam embankment after slide repair.

A key factor in the occurrence of the slide was the highly plastic clay *slopewash* (a naturally-occurring soil formed by weathering, erosion and down-slope movement) that blankets the hillside beneath the upstream slope of the dam. When the embankment was constructed, this material was dry and very strong. When it was wetted by filling the reservoir it softened dramatically and, since it was subjected to repeated cycles of shearing due to rise and fall of the reservoir, its strength was reduced to the residual value.[4] Shear strengths of the slopewash material, measured on specimens cut from undisturbed block samples obtained in a dry area downstream from the dam, are shown in Figure 2.

The slide was stabilized by constructing a berm against the upstream slope, as shown Figure 3. The factor of safety with the berm in place was evaluated in two steps:

1. Back analyses were performed to develop a computational model for the slope. A key element in this model was the strength of the highly plastic clay slopewash. The back analyses showed that the shear strength of the slopewash had been reduced to residual strength by wetting and shear displacement.

2. Forward analyses were performed to calculate the factor of safety with the berm in place. The forward analyses used the computational model developed through the back analyses, with the berm added to the cross-section as shown in Figure 3. The results of the analyses are summarized in Table 1.[4]

Olmsted Landslide Stabilization

To provide sufficient and reliable navigation capacity on the lower Ohio River, the U.S. Army Corps of Engineers is replacing Locks and Dams 52 and 53 with the proposed Olmsted Locks and Dam. This facility will include:

- Two locks, each 110 feet wide and 1,200 feet long, adjacent to the Illinois shore;
- A 2,200-foot wide navigable pass controlled by wicket gates; and,
- A 426-foot wide gated spillway.

Evidence of instability on the Illinois shore at the Olmsted site was first discovered in 1987 during the foundation investigation for the proposed locks and dam. On the basis of geomorphic features such as scarps, cracks, leaning trees and hummocky terrain, the approxi-

TABLE 1.
Computed Factors of Safety for the San Luis Dam Embankment

		Slopewash Strength		
Stage	**Phase & Condition of Slopewash**	***c* (ksf)**	**ϕ (degrees)**	**Computed *F***
Pre-failure	1. End of Construction (Slopewash Dry)	5.5	39	4.0
Pre-failure	2. Reservoir Full (Slopewash Soaked, Fully Softened)	0	25	2.0
Pre-failure	3. Drawdown (Slopewash Soaked, Fully Softened)	0	25	1.3
Failure	4. Drawdown (Slopewash Soaked, Residual)	0	15	1.0
Post-repair	5. Drawdown With Berm (Slopewash Soaked, Residual)	0	15	1.21

mate extent of unstable ground was mapped. The slide was found to be about 3,000 feet long.

In late May and early June 1988, a rapid drop in the river level from elevation 290 feet to elevation 283 feet took place over a 10-day period. In the lower portion of the river bank, near-vertical scarps up to 3 feet high developed 150 feet to 200 feet from the shoreline. Over the next month, the scarps and cracks propagated parallel to the river and eventually reached a total length of 3,100 feet. A cross-section through the landslide is shown in Figure 4.

Key factors in the occurrence of the slide were the low shear strength of the upper part of the McNairy I formation beneath the slope, and the artesian pressures in this formation. The McNairy I formation contains alternating layers of clay and fine sand varying from about an inch to about a foot in thickness. The position of the rupture surface (as shown in Figure 4) was established based on 12 inclinometers and 5 piezometer breaks. The rupture surface through the McNairy I was horizontal over a considerable distance. This orientation is consistent with shearing parallel to bedding, through clay layers. At the back of the slide, the rupture surface cut across bedding, passing through both clay layers and sand layers.

Different shear strengths were used for shear parallel to bedding in the McNairy I,

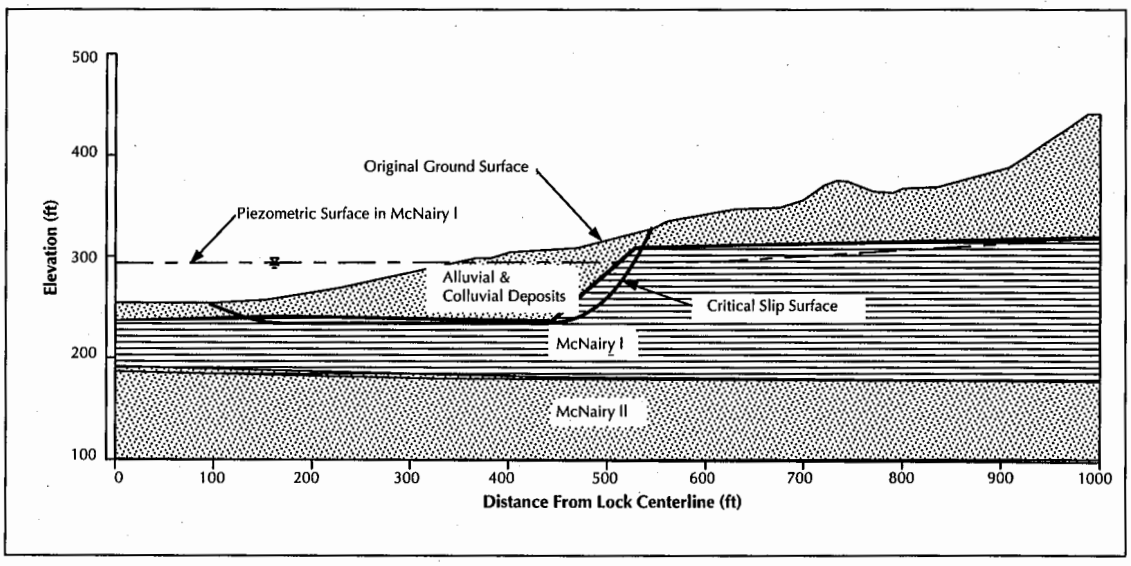

FIGURE 4. A cross-section through the landslide at Olmstead Locks and Dam on the Ohio River.

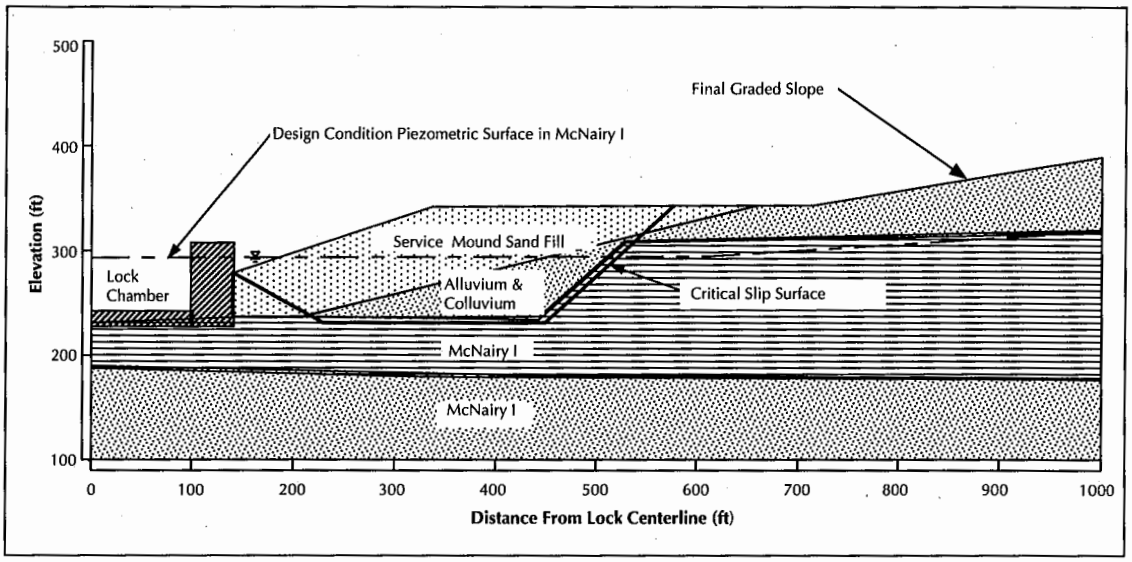

FIGURE 5. Long-term stability condition at Olmstead Locks and Dam on the Ohio River.

where the shear plane could pass entirely through clay and shear across bedding, and where shearing would necessarily involve both sand and clay. The McNairy I was assigned anisotropic shear strength values — with ϕ' equal to 10 degrees where the slip surface was within 5 degrees of horizontal and ϕ' equal to 25 degrees where the slip surface was inclined more than 5 degrees from horizontal. These values were thought to represent reasonable estimates of the residual strengths for bedding shear and cross-bedding shear.[5]

The piezometric levels in the McNairy I had been observed in a number of piezometers before the landslide occurred, and it was possible from these observations to estimate the piezometric level at the time of sliding.[5] Because the horizontal permeability of the McNairy I was much higher than the vertical permeability, the piezometric surface was nearly horizontal from the head of the landslide to the toe, and there were significant artesian pressures in the McNairy I beneath the lower part of the slope.

The slide was stabilized by constructing a service mound sand fill, which formed a buttress between the slope and the lock wall, as shown in Figure 5.[6] The factor of safety with the service mound fill in place was calculated in two steps:

1. Back analyses were performed to develop a computational model for the slope. Key elements in this model were the anisotropic shear strength of the McNairy I, and the piezometric levels in the slope. Using the same computational model, factors of safety varying from 0.95 to 1.00 were computed for four sections through the 3,100-foot-long slide.[5]

2. Forward analyses were performed to calculate the factor of safety during construction and at the end of construction of the berm. The analyses showed that it would not be feasible to construct the berm in stages, with each stage consisting of placing an inclined layer of fill against the slope. However, if the fill was placed in horizontal layers between the lock wall and the slope, the computed factor of safety increased steadily as the elevation of the berm was raised, reaching 2.1 at the end of construction, with the cross-section shown in Figure 5. The results of the analyses are shown in Table 2.[6]

La Esperanza Abutment Drainage

La Esperanza Dam impounds a water supply reservoir on the Carrizal River in northern Ecuador. The spillway of the dam is founded on potentially unstable shale in the right abutment.[7] The shale is essentially horizontally bedded, and contains numerous randomly oriented joints, with an average spacing of a few

TABLE 2.
Computed Factors of Safety for the Illinois Shore at Olmstead

Condition	Computed *F*
1. Pre-construction Landslide, After Drop in River Level	0.99
2. Post-construction, With Lock & Stabilizing Berm in Place	2.1

inches to about 2 feet. A cross-section through the abutment and spillway is shown in Figure 6.

Minor slides occurred during the excavation of the slope below the spillway, calling attention to a weak layer of breccia (brecciated shale) at the base of the slope. It seems likely that the brecciated layer was formed by valley wall rebound as the valley was eroded in the shale. The slides that occurred during construction were shallow, and did not threaten the spillway. However, larger slides through the shale and breccia, extending further up the slope, could damage the spillway, and it was seen as necessary to install drains in the right abutment to improve its stability.

The types and locations of drains needed to improve stability were determined in two steps:

1. Back analyses were performed to develop a computational model for the slope. Key elements in this model were the shear strength of the breccia, the shear strength of the shale and the piezometric surface within the abutment. These quantities were estimated based on laboratory tests and correlations to estimate the shear strength of the breccia, and observed water levels in exploratory borings for the piezometric surface. The shear strength estimated for the jointed shale was based on engineering judgment, with consideration of the fact that the shale would have higher strength than the breccia.

The objective of the back analysis was to determine what piezometric surface would correspond to a factor of safety equal to 1.0. The piezometric level so determined (which is shown in Figure 6) was higher than the water levels observed in exploratory borings. It was concluded that the shear strengths and piezometric surface were reasonable and probably conservative because the computational model was adjusted to give a factor of safety of 1.0, although no deep slide had occurred.

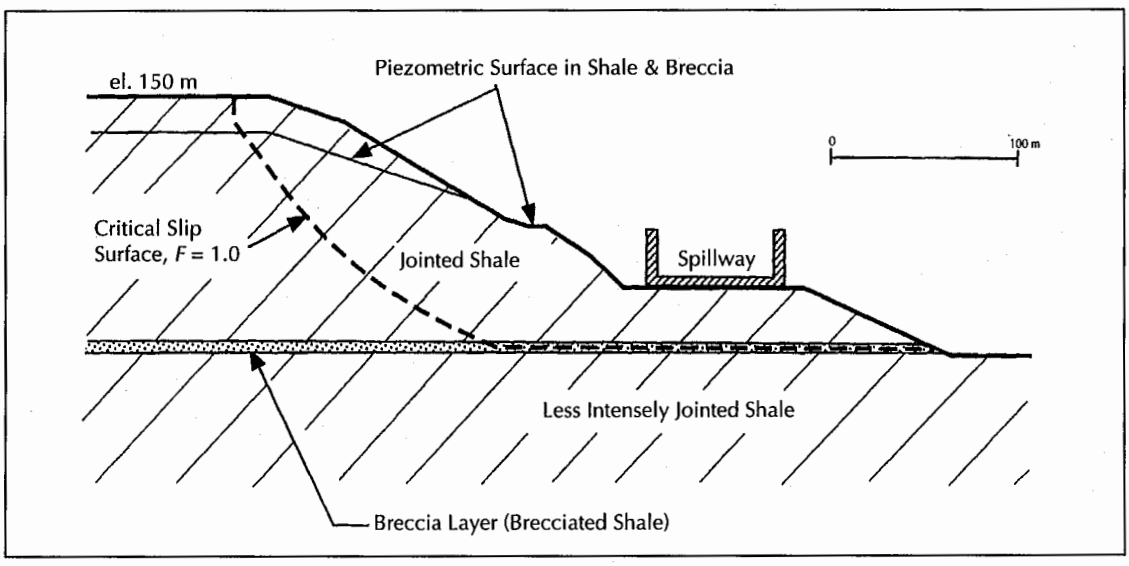

FIGURE 6. A cross-section through La Esperanza Dam's right abutment and spillway.

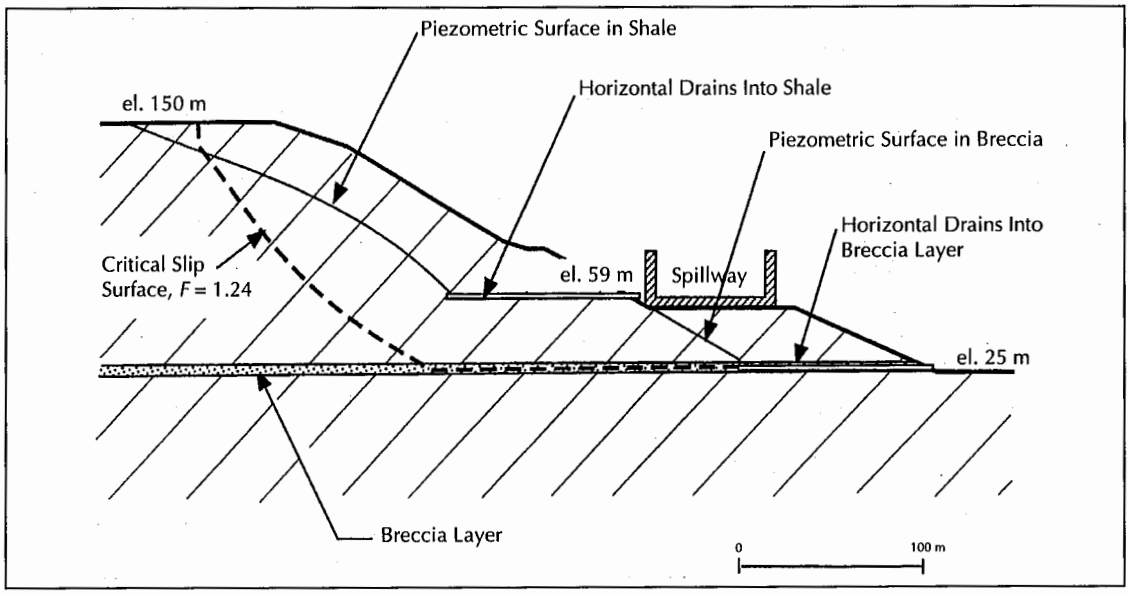

FIGURE 7. Drains in the right abutment of La Esperanza Dam.

2. Forward analyses were performed to determine the types and locations of drains needed to assure improved stability even if it should happen that impounding water in the reservoir, combined with exceptionally heavy rainfall, might cause the piezometric level to rise above the position shown in Figure 6.

As shown in Figure 7, it was assumed that the piezometric surface would rise to the ground surface in the upper part of the abutment, but would be lowered within the lower part of the abutment by installing two rows of horizontal drains at elevation 25 meters (in the breccia layer) and at elevation 59 meters (above the spillway). The computed factor of safety for the condition shown in Figure 7 is 1.24. The results of the analyses are shown in Table 3.

TABLE 3.
Computed Factors of Safety for the Right Abutment of La Esperanza Dam

Condition	Computed *F*
Initial Condition Shown in Figure 6 (Assumed to Be Marginally Stable)	1.0*
Drained Condition Shown in Figure 7, With Two Levels of Drains (90 m at el. 25 m & 120 m at el. 59 m)	1.24

* Achieved by adjusting shear strengths & piezometric surface.

The Effect of Assumptions in Back Analyses

As illustrated by these three examples, estimates and assumptions are always needed to fill gaps in knowledge when a computational model is developed by back analysis. However, adjusting the model so that the factor of safety is equal to 1.0 for the failure condition results in a reliable computational model in spite of the fact that it involves assumptions. If all of the elements of the model are consistent with the known facts, and if the computational model is used consistently in forward analysis, the assumptions will have little effect on the results of the forward analyses.

For example, the slope shown in Figure 8 was excavated by trimming back the original hill slope to a steeper configuration. (Please note that the slope shown in Figure 8 is hypothetical. Any resemblance to an actual failure is coincidental.) The shear strengths shown in the upper part of Table 4 were based on laboratory tests, and the piezometric level shown in

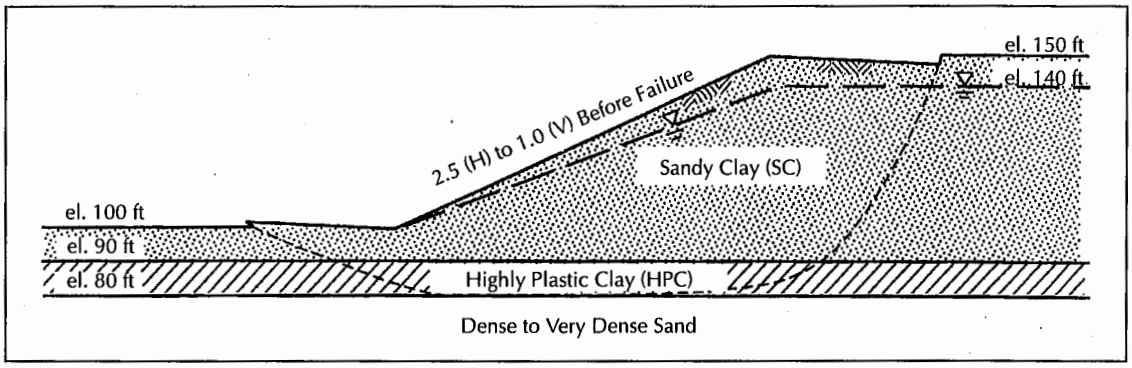

FIGURE 8. Failure in a hypothetical slope.

Figure 8 was based on water levels observed in exploratory borings.

For the conditions shown in Figure 8 and the strengths shown at the top row of Table 4, the computed factor of safety is 1.41, but the slope failed. Slope indicators installed after the failure showed that the rupture surface passed through the highly plastic clay. Piezometers installed after the failure showed that the piezometric surface in the highly plastic clay, and in the overlying layer of mixed sandy clays and clayey sands, was essentially as assumed in design.

It was proposed to stabilize the slope by installing vertical drain wells as shown in Figure 9, with horizontal drains drilled from the bottom of the slope to drain the water from these wells by gravity. It was estimated that these drains would draw down the average piezometric level along the line of drain wells to elevation 128 feet. This elevation is above the intersection of the drains with the wells, allowing for head losses associated with flow to the wells and through the drains. The factor of safety with the drains installed was calculated in two steps:

1. Back analyses were performed to develop a computational model for the slope. After reviewing the shear strengths for the layer of mixed sandy clays and clayey sands, it was concluded that the values used in the original analyses (shown in the top row in Table 4) were reasonable. Likewise, it was concluded that the piezometric levels used in the original analyses (shown in Figure 8) were reasonable. It seemed likely that the strength of the highly plastic clay might be lower than thought originally, and the friction angle for this layer was adjusted downward to achieve a factor of safety of 1.0 for the slope. It was found that, for a factor of safety of 1.0, a friction angle of 14 degrees

TABLE 4.
Original Design Analysis, Back Analysis & Redesign Analyses of a Hypothetical Slope

Analysis	SC Clay Strength	SC Unit Weight	HP Clay Strength	Piezometric Elevation	Computed *F*
Original Design Analysis	$c' = 300$ psf $\phi' = 35°$	$\gamma = 130$ pcf	$c' = 0$ $\phi' = 25°$	140 ft	1.41
Back Analysis	$c' = 300$ psf $\phi' = 35°$	$\gamma = 130$ pcf	$c' = 0$ $\phi' = 14°$	140 ft	0.99
Redesign Analysis	$c' = 300$ psf $\phi' = 35°$	$\gamma = 130$ pcf	$c' = 0$ $\phi' = 14°$	128 ft	1.16

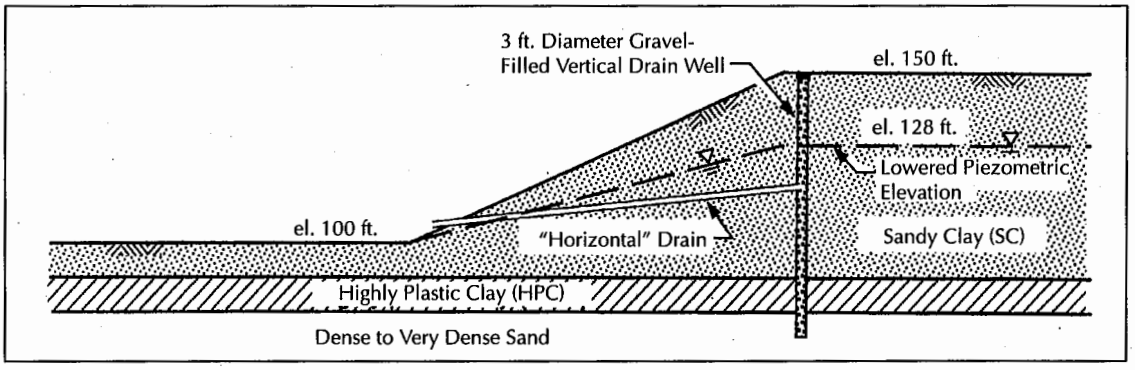

FIGURE 9. Drain wells to stabilize the hypothetical slope.

would be required in the highly plastic clay, as shown in the middle row of Table 4. This seemed reasonable because 14 degrees was approximately equal to the estimated residual friction angle of this material.

2. Forward analyses were performed for the conditions shown in Figure 9 in order to determine the factor of safety with the drains installed. The computed factor of safety, with the piezometric level at elevation 128 feet and ϕ' equal to 14 degrees in the highly plastic clay, was 1.16. The results of these analyses are summarized in the bottom row in Table 4.

In order to perform the back analyses, it was necessary to estimate (or assume) the strength of the sandy clay and the position of the piezometric surface. Because there are some uncertainties in these assumptions, it is reasonable to pose the following question: How would the computed factor of safety for the stabilized slope be affected, if different assumptions were used as the basis for the back analysis? To answer this question, the back analysis and forward analysis were repeated using different assumptions for the strength of the sandy clay.

First, the strength of the sandy clay was assumed to be much *higher* than assumed originally. When this assumption was made, the strength of the highly plastic clay required for a factor of safety equal to 1.0 in the back analysis was lower (as shown in the middle row in Table 5). However, the factor of safety calculated in the forward analysis was the same (*i.e.*, 1.16).

Next, the strength of the sandy clay was assumed to be much *lower* than assumed originally. When this assumption was made, the strength of the highly plastic clay required for a factor of safety equal to 1.0 in the back analysis was higher (as shown at the bottom row in Ta-

TABLE 5.
The Effect of Varying Sandy Clay Strength Used in the Back Analysis of a Hypothetical Slope

Analysis	Assumed SC Clay Strength	HP Clay Strength for $F = 1.0$ in Back Analysis	F With Piezometric Elevation at 128 Feet
Original	$c' = 300$ psf $\phi' = 35°$	$c' = 0$ psf $\phi' = 14°$	1.16
High SC Strength	$c' = 600$ psf $\phi' = 40°$	$c' = 0$ psf $\phi' = 9.6°$	1.16
Low SC Strength	$c' = 100$ psf $\phi' = 30°$	$c' = 0$ psf $\phi' = 18.2°$	1.16

ble 5). However, the factor of safety calculated in the forward analysis was again 1.16.

This exercise shows that the factor of safety calculated in the forward analysis is not strongly affected by the assumptions made as the basis for the back analysis. In this case, although the assumptions vary so widely as to stretch the bounds of credibility, the factor of safety calculated in the forward analysis remains the same because, in each case, the results of the back analysis are used consistently in the forward analysis.

Factor of Safety & Reliability

It stands to reason that, when back analysis can be used to reduce uncertainties in design, it would be justified to use smaller factors of safety than are used when shear strengths are based on laboratory tests or correlations. But, how much smaller factors of safety should be used? For example, should the value of 1.16 for the slope shown in Figure 9 be considered acceptable?

Answering this question in a vacuum is difficult. Different geotechnical engineers, drawing on different experiences, might well reach different conclusions. Furthermore, the consequences of another failure should be reckoned. If the consequences of failure are benign, a smaller factor of safety could be tolerated than if the consequences are severe financially, or if a failure would pose a threat to life.

Factors of safety used in geotechnical engineering are based on experience, and it appears that factors of safety that are conventionally used for slope stability are appropriate for conditions where shear strengths are based on laboratory tests, or are estimated based on correlations. When the computational model is based on back analysis, much uncertainty is eliminated, but the degree of uncertainty is not indicated by the value of factor of safety. Christian *et al.* showed that a high value for the factor of safety for a condition involving much uncertainty may involve greater risk than a low value for a condition involving little uncertainty.[8,9] To evaluate the importance of uncertainty and the level of risk that results from uncertainty, another measure of safety is needed. The probability of failure or, more precisely, the probability that the actual factor of safety is less than 1.0, is such a measure.

Probability of failure, as it is used in reliability theory, is the probability of an event occurring, sometimes called a "negative outcome." With regard to factor of safety calculations, the "event" is that the factor of safety is less than 1.0. In this context, then, what is called the probability of failure in reliability theory is the probability that the factor of safety could be less than 1.0, given the possible variations in the values of the quantities involved in its calculation.

Reliability as it is used in reliability theory is the probability of a "positive outcome." Reliability is the complement of the probability of failure. Thus, if there is a 0.5 percent probability that the factor of safety is less than 1.0, the reliability (the probability that the factor of safety is greater than 1.0) is 99.5 percent.

The probability that the factor of safety is less than 1.0 can be calculated using a simple method, with the same amount and quality of data as used in more traditional geotechnical engineering factor of safety calculations. The calculated value of this probability should be viewed as no more accurate than the results of most geotechnical analyses. Even so, it provides a useful measure of the combined effects of uncertainties in the values of the parameters involved in computing the safety factor.

Several sources have outlined simple and effective procedures for estimating the probability that the factor of safety is less than 1.0.[10-12] Using the methods they described, this probability can be calculated following these steps:

1. Estimate the *most likely values* (MLVs) of the parameters required for the analysis. These values can be estimated using the same types of data ordinarily used for deterministic analysis, and applying engineering judgment in the same way. However, a conservative bias should not be built into the parameter values. Most likely values are appropriate.

2. Estimate the standard deviations of the parameters. Where too few data are available to compute the standard deviation by formal statistical methods, reasonable values can be estimated using the *three-sigma rule* and engineering judgment.

3. Calculate the factor of safety using the most likely values of the parameters. This

value of the factor of safety is the most likely value of the factor of safety (or, F_{MLV}).

4. Calculate factors of safety with the value of each of the variables in turn *increased* by one standard deviation from its most likely value, and the other variables assigned their most likely values. These factors of safety are denoted as $(F^+)_i$, where i denotes the parameter whose value is increased. There are N values of $(F^+)_i$, where N is the number of parameters involved in the calculation of F.

Also calculate factors of safety with the value of each of the variables in turn *decreased* by one standard deviation from its most likely value, and the other variables assigned their most likely values. These factors of safety are denoted as $(F^-)_i$. There are N values of $(F^-)_i$, where N is the number of parameters involved in the calculation of F.

5. Use the Taylor series method to compute the standard deviation of the factor of safety by using Equation 1.

6. Determine the probability of the factor of safety being less than 1.0 using Table 6.

Computing Probabilities That the Factor of Safety May Be Less Than 1.0

Table 6 is convenient because it shows values of the probability of a factor of safety of less than 1.0 related to F_{MLV} and σ_F. Its shortcoming is that only approximate values of the probability of a factor of safety of less than 1.0 can be determined for values of F_{MLV} and σ_F that are intermediate between the values listed in the table. It may sometimes be desirable to have a means of computing more precise values of the probability of a factor of safety of less than 1.0.

The key to computing more precise values of the probability of a factor of safety of less than 1.0 is to compute the value of the log normal reliability index, β_{LN}, using Equation 2.

When β_{LN} has been computed, the value of the probability of a factor of safety of less than 1.0 can be determined accurately in either of two ways:

1. Use tables of the standard cumulative normal distribution function, which can be found in many textbooks on probability and reliability. For example, the value of β_{LN} computed for the original design analysis of the hypothetical slope, corresponding to F_{MLV} equal to 1.41 and σ_F equal to 0.204, is 2.31. The standard cumulate normal distribution function (the reliability) corresponding to a β_{LN} equal to 2.31 is 0.9896.[13] The probability of a factor of safety of less than 1.0 is one minus the reliability (or equal to 1.0 minus 0.9896), which is 0.0104.

2. Use the built-in function NORMSDIST in Microsoft Excel. The argument of this function is the reliability index, β_{LN}. In this spreadsheet program, under "Insert Function," "Statistical," choose "NORMSDIST," and type the value of β_{LN}. For example, for β_{LN} equal to 2.31, the result is 0.989556, which corresponds to the probability of a factor of safety less than 1.0 of 0.0104. Table 6 was developed using this function.

These methods assume a log-normal distribution of factor of safety. The log-normal distribution was selected because the central limit

EQUATION 1.

$$\sigma_F = \sqrt{\left(\frac{\Delta F_1}{2}\right)^2 + \left(\frac{\Delta F_2}{2}\right)^2 + \ldots \left(\frac{\Delta F_N}{2}\right)^2}$$

where:

σ_F = standard deviation of factor of safety

$\Delta F_1 = \{(F^+)_1 - (F^-)_1\}$, $\Delta F_2 = \{(F^+)_2 - (F^-)_2\}$, etc.

EQUATION 2.

$$\beta_{LN} = \frac{\ln\left(\frac{F_{MLV}}{\sqrt{1+(\sigma_F / F_{MLV})^2}}\right)}{\sqrt{\ln\left(1+(\sigma_F / F_{MLV})\right)^2}}$$

where:

β_{LN} = log normal reliability index,

σ_F = standard deviation of factor of safety (see Equation 1), and

F_{MLV} = most likely value of factor of safety.

TABLE 6.
Probabilities That a Factor of Safety Is Smaller Than 1.0

F_{MLV}	Standard Deviation of Factor of Safety																		
%	0.02	0.04	0.06	0.0675	0.08	0.1	0.12	0.14	0.16	0.2	0.2041	0.25	0.3	0.4	0.5	0.6	0.8
1.10	0.000	0.461	4.259	6.375	10.099	15.740	20.557	24.557	27.882	33.057	33.503	37.733	41.219	46.245	49.871	52.727	57.124
1.12	0.000	0.080	1.830	3.201	6.022	10.986	15.689	19.825	23.384	29.079	29.576	34.322	38.250	43.887	47.899	51.018	55.746
1.14	0.000	0.010	0.685	1.443	3.330	7.314	11.590	15.630	19.261	25.286	25.822	30.987	35.310	41.251	45.912	49.293	54.353
1.163	0.000	0.001	0.185	0.501	1.525	4.304	7.840	11.527	15.050	21.210	21.772	27.285	31.988	38.805	43.614	47.291	52.735
1.18	0.000	0.000	0.062	0.207	0.797	2.779	5.693	8.999	12.328	18.423	18.993	24.664	29.590	36.808	41.913	45.804	51.529
1.20	0.000	0.000	0.015	0.065	0.342	1.578	3.771	6.553	9.559	15.417	15.982	21.727	26.850	34.480	39.912	44.047	50.101
1.25	0.000	0.000	0.000	0.002	0.027	0.295	1.125	2.602	4.583	9.275	9.773	15.204	20.480	28.823	34.953	39.651	46.499
1.30	0.000	0.000	0.000	0.000	0.001	0.037	0.254	0.843	1.883	5.062	5.443	10.001	14.965	23.516	30.131	35.297	42.872
1.35	0.000	0.000	0.000	0.000	0.000	0.003	0.042	0.219	0.653	2.482	2.736	6.148	10.435	18.687	25.534	31.046	39.253
1.40	0.000	0.000	0.000	0.000	0.000	0.000	0.005	0.044	0.189	1.083	1.230	3.511	6.916	14.433	21.247	26.958	35.674
1.41	0.000	0.000	0.000	0.000	0.000	0.000	0.003	0.031	0.144	0.904	1.033	3.109	6.329	13.657	20.433	26.166	34.966
1.50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.009	0.140	0.172	0.897	2.569	7.840	13.855	19.494	28.766
1.60	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.010	0.014	0.160	0.743	3.707	8.270	13.256	22.399
1.70	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.019	0.163	1.501	4.474	8.420	16.783
1.80	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.026	0.512	2.171	4.962	12.057
1.90	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.003	0.145	0.936	2.695	8.276
2.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.034	0.355	1.340	5.407
2.20	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.034	0.248	1.964
2.40	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.030	0.563
2.60	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.124
2.80	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.020
3.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002

theorem indicates that the normal distribution is the most likely form of distribution of a quantity resulting from the addition and subtraction of many randomly distributed values.[14] It follows that the log-normal distribution is the most likely form of distribution of a quantity that results from many multiplication and division operations, because multiplication and division are equivalent to adding and subtracting logarithms.

The Three-Sigma Rule

The three-sigma rule is very useful for estimating the value of the standard deviation of a parameter when too little data is available to calculate the standard deviation using formal statistical methods.[13] The three-sigma rule relies on the fact that, as shown in Figure 10, 99.73 percent of all of the values of a normally distributed variable lie within plus or minus three standard deviations ($\pm 3\sigma$) of the mean (or most likely value). For practical purposes this is 100 percent (or all values). This fact can be used to estimate the standard deviation, as follows:

1. Estimate the highest conceivable value (HCV) and the lowest conceivable value (LCV) of each variable, using all available data, correlations and engineering judgment. A conscious effort should be made to make the range wide because the objective is to encompass all conceivable values. Controlled experiments have shown that there is a tendency to estimate a range of values that is narrower than the actual range.[15]
2. Compute the value of standard deviation by subtracting the LCV from the HCV and dividing the result by 6.

The three-sigma rule makes it possible to overcome an often-encountered barrier to the use of reliability methods — insufficient data. Using it, engineers can estimate values of standard deviation on the basis of the same kinds of judgments required to select parameter values for deterministic analyses.

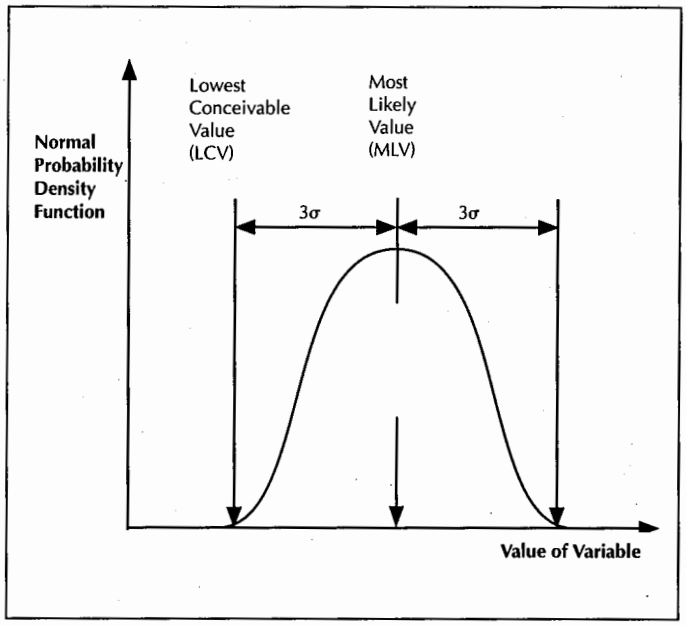

FIGURE 10. The basis for the three-sigma rule.

Probability of Factor of Safety Less Than 1.0 for the Hypothetical Slope

The method outlined by Wolff and the U.S. Army Corps of Engineers has been applied to the hypothetical slope shown in Figures 8 and 9 to evaluate the reliability of the original design, and to evaluate the reliability of the redesign based on back analysis.[10-12] The results are shown in Tables 7 and 8.

As noted previously, and shown in the top row of Table 7, the factor of safety calculated in the original design, using the most likely values of all of the parameters, was 1.41. There were four major sources of uncertainty in the original design — the strength of the sandy clay, the unit weight of the sandy clay, the strength of the highly plastic clay and the piezometric elevation. The highest and lowest conceivable values and corresponding standard deviations for each of these quantities are shown in Table 7, together with values of F^+ and F^- corresponding to varying their values by plus or minus one standard deviation from the most likely values. Also shown are the values of ΔF for each variable, the standard deviation of the factor of safety computed by subtracting the LCV from the HCV and dividing the result by 6, and the probability of failure from Table 6. The location in Table 6 corre-

TABLE 7.
Reliability Analysis of the Hypothetical Slope — Original Design

With all quantities assigned their most likely values: F_{MLV} = 1.41			
Variable Quantity	**Parameter Values**	**Factors of Safety**	
Sandy Clay Strength			
Most likely value	c' = 300 psf, ϕ' = 35°		
Highest conceivable value	c' = 1000 psf, ϕ' = 45°	F^+ = 1.55*	
Lowest conceivable value	c' = 0 psf, ϕ' = 25°	F^- = 1.31*	ΔF = 0.24**
Standard deviation	σ_c = 167 psf, σ_ϕ = 3.3°		
Sandy Clay Unit Weight			
Most likely value	γ = 130 pcf		
Highest conceivable value	γ = 145 pcf	F^+ = 1.46	
Lowest conceivable value	γ = 115 pcf	F^- = 1.38	ΔF = 0.08**
Standard deviation	σ_γ = 5 pcf		
Highly Plastic Clay Strength			
Most likely value	c' = 0, ϕ' = 25°		
Highest conceivable value	c' = 0, ϕ' = 35°	F^+ = 1.54	
Lowest conceivable value	c' = 0, ϕ' = 15°	F^- = 1.29	ΔF = 0.25**
Standard deviation	σ_c = 0, σ_ϕ = 3.3°		
Piezometric Elevation			
Most likely value	el. = 140 ft		
Highest conceivable value	el. = 150 ft	F^+ = 1.52	
Lowest conceivable value	el. = 120 ft	F^- = 1.32	ΔF = 0.20**
Standard deviation	σ_{el} = 5 ft		
Standard deviation of factor of safety (see Equation 1) = 0.2041			
Probability of factor of safety less than 1.0 (see Table 6) = 1.03%			

*F^+ is calculated with both c' and ϕ' increased by one standard deviation, and F^- is calculated with both c' and ϕ' decreased by one standard deviation.
**$\Delta F = F^+ - F^-$

sponding to the original design is highlighted. The probability of the factor of safety being less than 1.0 is 1.03 percent.

As noted previously, and as indicated in the top row of Table 8, the factor of safety calculated in the redesign, based on the results of the back analysis, was 1.16. All of the parameters except the piezometric level are determined reliably by the back analysis and are, therefore, not subject to significant uncertainty. This being the case, the standard deviation of the redesign factor of safety is much smaller than for the original design factor of safety. The probability of the factor of safety being less than 1.0 is 0.5 percent. The location in Table 6 corresponding to the redesign is highlighted.

Although the factor of safety for the redesign is lower than the factor of safety for the original design (a factor of safety equal to 1.16 as compared to 1.41) the probability of the redesign factor of safety being less than 1.0 is smaller (0.5 percent as compared to 1.0 percent). This difference is a direct result of the fact that the redesign was based on back analysis, which eliminates much uncertainty.

It is not the purpose of this example to indicate that the redesign condition analyzed above represents a sufficiently reliable long-term condition for the slope. In terms of the probability of a factor of safety less than 1.0, it is not much different from the original design, and it might be desirable to increase the factor of safety to a value greater than 1.16, and reduce the value of the probability of failure even more, by additional stabilizing measures. For example, if the spacing between

TABLE 8.
Reliability Analysis of the Hypothetical Slope Redesign

With all quantities assigned their most likely values: F_{MLV} = 1.16 (values are shown in Table 4)			
Variable Quantity	**Parameter Values**	**Factors of Safety**	
Piezometric Elevation			
Most likely value	el. = 128 ft		
Highest conceivable value	el. = 140 ft	F^+ = 1.23	
Lowest conceivable value	el. = 110 ft	F^- = 1.10	ΔF = 0.13**
Standard deviation	σ_{el} = 5 ft		
Standard deviation of factor of safety (see Equation 1) = 0.0675			
Probability of factor of safety less than 1.0 (see Table 6) = 0.5%			

**$\Delta F = F^+ - F^-$

wells was reduced so that the average piezometric level was lowered by an additional 3.0 feet, the factor of safety would increase from 1.16 to 1.20, and the value of probability would decrease to 0.05 percent.

The consequences of failure and the cost of stabilization are important factors in determinating the acceptable level of risk. These factors are beyond the scope of this hypothetical exercise. It is clear, however, that being able to compute the probability that the factor of safety will be less than 1.0 (in addition to computing the most likely value of the factor of safety) provides a more complete basis for making this judgment.

Conclusions

The preceding examination of back analysis, factor of safety and reliability supports these conclusions:

- Back analysis can be used to develop highly reliable computational models of slopes. It can be used where slopes have failed, such as at the San Luis Dam and Olmsted Locks and Dam, and where slopes have not failed, such as at La Esperanza Dam.
- When the results of back analysis are used for the design of slope stabilization measures, they should be used consistently with the assumptions made in the back analysis. While assumptions are inevitably required to develop computational models, adjusting the elements of the model through back analysis results in a model that is not significantly affected by the assumptions, provided that they are reasonable and fit all the known facts.
- As the hypothetical example described above clearly indicates, it is justified to use lower-than-conventional factors of safety when the computational model can be based on back analysis, because back analysis reduces slope failure risk.
- Reliability theory is a very useful complement to conventional deterministic analyses. Using the Taylor series method, it is possible to compute the probability of the factor of safety being less than 1.0 through the application of simple formulas. Computing both the most likely value of the factor of safety and the probability that the factor of safety could be less than 1.0 affords a better basis for judging analytical results than does computing only the factor of safety.
- The effort involved in computing the probability of the factor of safety being less than 1.0 is not great. Each of the additional computations required only involves changing the value of one variable and repeating the calculations. As compared with conventional deterministic

analyses, using the Taylor series method to compute the probability of a factor of safety of less than 1.0 may add 5 percent to the effort, and 50 percent to the value of the analysis.

- The three-sigma rule provides a very useful technique for overcoming the frequently encountered problem of too few data to compute the standard deviation by formal statistical methods. Using the three-sigma rule, it is possible to estimate standard deviations by means of the same combination of data, correlations and engineering judgment that are involved in all geotechnical engineering analyses.

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