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### PAPERS AND DISCUSSIONS



### OF GENERAL INTEREST



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JOURNAL OF THE

# BOSTON SOCIETY OF CIVIL ENGINEERS

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## MECHANICS OF HYDRAULIC-FILL DAMS

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(Presented at a meeting of the Boston Society of Civil Engineers held on May 16, 1934)

### INTRODUCTION

THE chief distinguishing characteristic of a hydraulic-fill dam is the automatic segregation, by means of flowing water, of the constituent soil particles into a fine-grained central core and a coarse-grained outer shell. The core is intended to furnish water-tightness, while the purpose of the shell is to provide stability. This sharp distinction in function, coupled with the fact that large quantities of sluicing water are trapped in the core during and after construction, gives rise to a number of problems in soil mechanics quite different from those associated with other types of structures.

The writer proposes to discuss three of these problems which may be considered fundamental: first, the stability of the shell against bursting by the internal pressure of the core; second, the rate of consolidation of the core; and third, the seepage of water through the core.

Analyses of these problems were included in a paper previously published.† Additional studies have clarified a number of points and have resulted in definite advances over the methods previously sug-

* Associate Professor of Soil Mechanics, Massachusetts Institute of Technology, Cambridge, Mass.

† "Hydraulic-Fill Dams." Proceedings, International Commission on Large Dams, World Power Conference, Paris, 1933.

gested. While there is still considerable room for improvement, it is believed that the principles thus far developed afford a clearer understanding of the basic mechanics than has hitherto been possible.

## Stability

During construction, and frequently for some time after completion, a hydraulic-fill core exhibits the characteristic properties of a liquid, due to the presence of excess water in the voids. In consequence, the principal criterion of the stability of the structure as a whole is that the shearing strength of the shell must be great enough in all parts to withstand the shearing stresses induced by the core pressure.

The shell is composed predominantly of a mixture of sand and gravel, which is a granular material with practically no cohesion. Hence its strength depends only upon its own weight and its angle of internal friction. The basic condition of stability in a material of this type is that the resultant stress on any plane within the mass must not have an inclination with the normal to the plane greater than the angle of internal friction of the material.

In order to arrive at a general solution it is necessary to replace the irregular outlines of core and shell by straight lines, as indicated in Fig. 1. The actual outline shown is that of the Germantown Dam, in Ohio. It is apparent that the actual section is slightly stronger than the idealized section, with the weakest points at the two upper berms. The approximation involved is quite small, and is justified by the gain in geometrical simplicity. It can be shown, furthermore, that the idealized section is equally strong at all elevations; hence the results obtained are independent of dimensions.

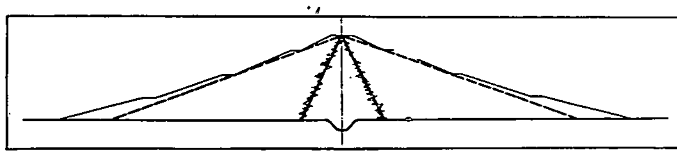

Fig. 1

The variables involved are five in number: the slope of the core, the outer slope, the weight per unit volume of the core, the weight per unit volume of the shell, and the angle of internal friction of the shell material. The unit weights enter as a ratio, so that only four variables need be included in the final result.

The following notation will be used:

$A$ = cotangent of angle of core slope with horizontal.
$B$ = cotangent of angle of internal friction of shell material.
$C$ = cotangent of angle of outer slope with horizontal.
$R$ = ratio of unit weight of core to unit weight of shell.

Fig. 2 is a schematic representation of these quantities. It will be observed that $A$ and $C$ are the usual slope constants, representing the ratio of horizontal run to vertical rise. Thus an outer slope of 2.5 to 1 corresponds to a value of $C$ of 2.5. It is also evident that all the variables are abstract numbers, so that the result is valid for any system of units.

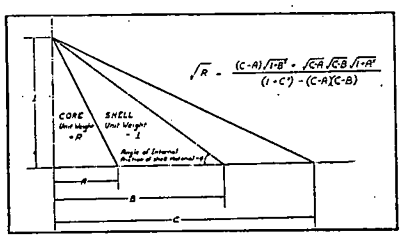


FIG. 2

For a section just on the point of failure, the variables are related as follows:

$$\sqrt{R}=\frac{(C-A)\sqrt{1+B^2}+\sqrt{C-A}\,\sqrt{C-B}\,\sqrt{1+A^2}}{(1+C^2)-(C-A)\,(C-B)}.^*$$

This equation is obviously a very simple one to handle in computations. Unfortunately, it does not lend itself to a graphical representation unless one of the variables is kept constant. To assist in visualizing the relations obtained, the values in Fig. 3 have been computed for $B=1.2$; that is, for a shell material with an angle of internal friction of a little less than 40°. The curves show clearly the important influence of the core slope. For example, if the weights of core and shell were equal, a core with a slope of 0.4 horizontal to 1 vertical would be just in equilibrium if the outer slope were as steep as 2 to 1; whereas a 45° core would require an outer slope of 3.8 to 1.

* A derivation of this equation is given in the Appendix.

Since the relation between the variables is independent of actual dimensions, it lends itself readily to an investigation of its validity by means of model tests. Mr. D. W. Taylor, of Massachusetts Institute of Technology, undertook an investigation of this kind, and obtained

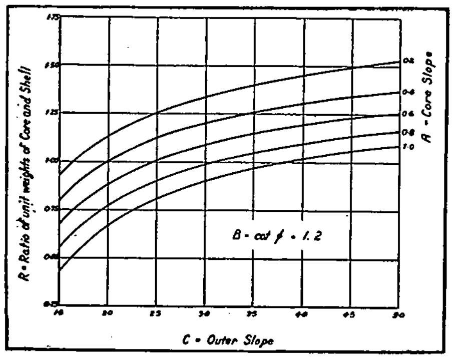


FIG. 3

results which corroborated the theoretical analysis. Model dams were constructed with a shell of sand and a core of a heavy liquid, — carbon tetrachloride. The sections were designed to fail in a certain definite location in accordance with the theory, and it was found that failure

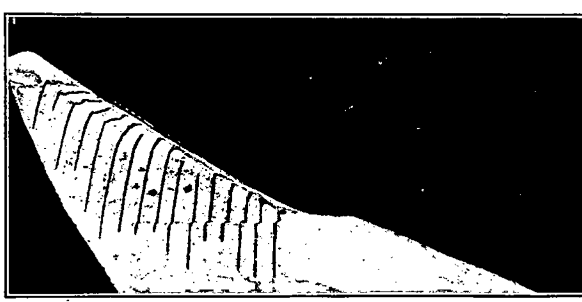

FIG. 4

actually took place in the manner anticipated. Fig. 4 is a photograph of a section removed from the center of a model after the sand had been impregnated with gelatine to keep it intact. The theory showed that failure ought to take place along a nearly horizontal plane through the berm. The strips of colored sand, originally vertical and continuous, show distinct breaks along a line which corresponds to the theoretical plane of failure as closely as measurements can indicate.

It would appear, then, that the theoretical solution reflects actual stress conditions in the shell in a satisfactory manner. From a practical standpoint, this means that the effects of changes in any or all of the variables can be studied easily and accurately, so that a design can be carried through on a more nearly rational basis than was formerly possible.

As far as measurements of the physical characteristics of the soils are concerned, the unit weights of core and shell can be estimated with a reasonable degree of accuracy, but the determination of the angle of internal friction of the shell material is a somewhat more difficult problem. Shearing machines * have been used successfully for sands and finer soils, but relatively little information has been obtained for coarser materials. A noteworthy attempt in this direction was made by Mr. H. H. Hatch in connection with soil studies on the Cobble Mountain Dam.† The results obtained offer interesting indications of the differences in strength encountered in various materials, and it is hoped that opportunities for further investigations may soon arise. Fortunately, however, sufficient is known at present about the physical behavior of cohesionless, granular soils so that, with the aid of the theoretical development, obviously weak designs can be avoided, and the shearing stresses in the shell can be kept within conservative limits. The accumulation of more precise data will permit a somewhat more advantageous utilization of the strength of coarse-grained mixtures than would now be justified.

The foregoing analysis gives rise to a number of interesting conclusions with regard to stability. The more obvious indications are that a relatively narrow core and a dense, heavy shell are to be preferred. The shell should be as nearly free from fine material as possible, for two reasons: water draining from the core which becomes trapped within the shell will tend to exaggerate the internal pressures which the shell must resist; and the presence of appreciable quantities of soft silt and clay in the shell will tend to reduce its shearing strength. In consequence, the writer is inclined to favor the true hydraulic-fill method, in which the soil is thoroughly broken up and mixed with water before deposition, rather than the semi-hydraulic-fill, in which

* See the writer's paper, "Soil Mechanics Research." Transactions, American Society of Civil Engineers, 1933.

† "Tests for Hydraulic-Fill Dams." Proceedings, American Society of Civil Engineers, October, 1932.

the soil is dumped on the embankment and then sluiced toward the core pool.

The writer is also of the opinion that too much emphasis has been placed upon the desirability of rapid consolidation of the core, and too little upon other equally important factors. Considering the uncertainties in drainage and pressure conditions, it seems advisable, in the interests of safety, to design the shell strong enough to resist the full liquid pressure of the core. If this is done, it is a matter of small moment whether the core consolidates rapidly or slowly. It is important, of course, to be able to estimate the rate of consolidation; but there should be no cause for alarm if this rate happens to be rather slow. Above all, there seems to be very little justification for attempting to obtain the dubious advantages of rapid consolidation by sacrificing water-tightness or wasting good material.

## Consolidation

The rate at which excess water can escape from a mass of fine-grained soil under pressure is governed by two basic factors: the drainage conditions and the stress system in and around the soil mass; and the physical characteristics of the mass itself, particularly its compressibility and its permeability. Analysis of the first factor involves the formulation of an equation which expresses the conditions of stress and drainage in general terms by fulfilling the fundamental consolidation equation and satisfying appropriate boundary conditions. Determinations of the physical characteristics are dependent upon actual measurements of the amount and rate of compression of the soil under various loads.

In the case of a hydraulic-fill core, the governing conditions are that water is free to escape along the two inclined faces of the core, and that initially the stress in the core varies uniformly from zero at the top to a maximum at the bottom. The latter statement would be strictly correct only if the dam were constructed instantaneously; but here, as in other consolidation problems, it is necessary to investigate the instantaneous case first, later introducing such modifications as are required to take into account the time of construction.

The problem, then, is to find an equation expressing the rate of consolidation of a mass of soil having a cross section in the shape of an isosceles triangle, with any desired base angle, and subjected to uniformly varying initial stress. The writer has recently worked out a solution for this case, with the limitation that the base angle must be

45°. The more general solution, applicable to any base angle, has not as yet been found; but the investigations on the 45° case have led the writer to believe that such a solution is not impossible.

Fortunately, the present situation is by no means as unsatisfactory as might be supposed. The base angle of the core will be, in general, somewhere between 45° and 90°. The 45° case has been worked out, and the vertical case can be shown to be identical with Terzaghi's solution for a layer of soil drained on both faces and subjected to uniform stress. Consequently, the limits between which the general solution must lie are established; and an approximation to the general case may be obtained by a simple interpolation.

A discussion of the theoretical development would be too long to be included here. The results are as follows:

For a vertical core —

$$Q = 1 - \frac{8}{\pi^2} \sum_{m=0}^{m=\infty} \frac{1}{(2m+1)^2} \epsilon^{-\frac{(2m+1)^2\pi^2}{4}\frac{Kt}{b^2}}$$

For a core with 45° slopes —

$$Q = 1 - \frac{24}{\pi^4} \left[ \sum_{m=0}^{m=\infty} \sum_{n=0}^{n=\infty} \frac{2}{(2m+1)^2(2n+1)^2} \epsilon^{-[(2m+1)^2+(2n+1)^2]\frac{\pi^2}{2}\frac{Kt}{b^2}} + \sum_{n=0}^{n=\infty} \frac{1}{(2n+1)^4} \epsilon^{-(2n+1)^2\pi^2\frac{Kt}{b^2}} \right]$$

In these equations, $m$ and $n$ take successive integral values from 0 to $\infty$; $\epsilon$ is the Napierian base; $t$ is the time elapsed from the beginning of the consolidation process; $b$ is half the width of the core at the bottom; and $K$ is a coefficient expressing the consolidation characteristics of the soil, equal to the product of the coefficient of permeability and the bulk modulus. The result, $Q$, is the percentage consolidation, expressed as a ratio, at time $t$; $Q$ varies from 0, for the unconsolidated initial state at time 0, to 1, or 100 per cent, for the completely consolidated state, at time $\infty$.

Inspection of the units of the fraction $\frac{Kt}{b^2}$ which contains all the variables, shows it to be a dimensionless ratio. Therefore if a "time factor," $T$, is defined by —

$$T = \frac{Kt}{b^2}$$

and substitution is made in the equations, they will consist entirely of abstract numbers. By assigning values to $T$ and computing the corresponding values of $Q$, a curve of $Q$ against $T$ for each case can be obtained which is valid for all possible combinations of $K$, $b$, and $t$; that is, valid for all soils, for masses of all sizes, and for any consistent system of units. Once this general curve has been obtained, the equation is no longer needed; by a change of scale along the horizontal axis the general curve of $Q$ against $T$ can be transformed into a particular curve of $Q$ against actual time, $t$, for any given particular values of $K$ and $b$.

The two general curves are plotted in Fig. 5. For convenience, values of $T$ for the two cases are given in Table I.

TABLE I

| Q | T | |
|---|---|---|
| | 45° Core | 90° Core |
| 0.00 | 0.000 | 0.000 |
| 0.10 | 0.002 | 0.008 |
| 0.15 | 0.004 | 0.018 |
| 0.20 | 0.007 | 0.031 |
| 0.25 | 0.010 | 0.049 |
| 0.30 | 0.015 | 0.071 |
| 0.35 | 0.020 | 0.096 |
| 0.40 | 0.027 | 0.126 |
| 0.45 | 0.034 | 0.159 |
| 0.50 | 0.042 | 0.197 |
| 0.55 | 0.052 | 0.238 |
| 0.60 | 0.063 | 0.287 |
| 0.65 | 0.076 | 0.347 |
| 0.70 | 0.092 | 0.405 |
| 0.75 | 0.110 | 0.477 |
| 0.80 | 0.132 | 0.565 |
| 0.85 | 0.162 | 0.684 |
| 0.90 | 0.203 | 0.848 |
| 0.95 | 0.272 | 1.127 |
| 1.00 | ∞ | ∞ |

The general curve for a core with a base angle between 45° and 90° should lie between these two limiting curves. Without a solution for the intermediate case it is impossible to say just what position the curve would take. However, when attempting a solution for the more general case, the writer found that the space relation was the difficult one to satisfy; the time function came out readily enough. In this time function the exponent of $\epsilon$ was of a form similar to that found in the equations above cited, except that it contained the cotangent of the base angle. While this relation would be modified somewhat by the form of the unknown space function, it suggests at once a very simple method of interpolation between the two limiting curves. For a 90° core the cotangent is 0, and for a 45° core it is 1. For an intermediate case, a direct linear interpolation on the basis of the value of the cotangent is suggested. This also ties in with the notation in the previous section, where the cotangent was denoted by $A$. Therefore the 45° curve of Fig. 4 would correspond to $A=1$, and the 90° curve to $A=0$. For a value of $A=0.5$, the general curve would lie halfway, measuring horizontally, between the limiting curves; and similarly for other intermediate values of $A$.

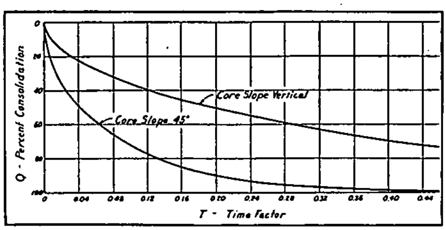


FIG. 5

The procedure for estimating the rate of consolidation may now be summarized as follows: knowing the shape of the core, its value of $A$ is found and used to interpolate between the two limiting curves to get a general curve of $Q$ against $T$ applicable to that particular shape; the dimensions of the core determine the half-width, $b$; from tests on the core material the value of the coefficient $K$ is obtained; and from the relation $T=\frac{Kt}{b^2}$, the $T$-scale of the diagram is transformed into a scale of actual time, in months or years.

The resulting curve shows the rate at which consolidation would proceed with time, provided the dam were constructed instantaneously. Since in actuality the construction occupies an appreciable period, some consolidation will take place before completion.

It would be very difficult to analyze this effect, although some idea of the situation could be obtained by a method of successive approximations. On the other hand, cores are frequently so impervious that the construction period is a relatively short time. Under these circumstances, a reasonable result can be obtained by shifting the instantaneous curve to the right a distance equal to half the construction period, and sketching in the portion of the curve from the beginning to the end of this period.

The only data available to check the theoretical concept are those obtained from studies on the core material of the Germantown Dam of the Miami Conservancy District. Tests of the physical characteristics of undisturbed core samples, combined with computations of the weights of overlying materials, indicated that the core was about 25 per cent consolidated when the samples were removed.

The theoretical rate of consolidation can be worked out in accordance with the foregoing rules. The core was designed with slopes of 1 horizontal to 2 vertical, corresponding to a value of $A$ of 0.5. Then the general consolidation curve would lie halfway between the two curves of Fig. 5. Interpolating on this basis in Table I, the values of $T$ corresponding to this particular shape are as shown in the second column of Table II.

The relation between $T$ and the actual time, $t$, can be found by using the proper values of $K$ and $b$. Consolidation tests on the core material showed that the average value of $K$ was 0.0183 square centimeters per minute. The width of the core at the base is 110 feet, so that $b$ is 55 feet, or 1,690 cm. Then —

$$T=\frac{Kt}{b^2}=\frac{0.0183}{(1690)^2}\,t \qquad \text{(with } t \text{ in minutes)}$$

or

$$t=297\ T \qquad \text{(with } t \text{ in years).}$$

The construction period, about two years, is relatively short in this case, so that its effect may be approximated by moving the curve one year to the right, and sketching in the section between $t=0$ and $t=2$ years. The relation between $t$ and $T$ is then —

$$t=297\ T+1 \qquad \text{(for values of } t \text{ greater than 2).}$$

In this fashion the values of $t$ in the third column of Table II are obtained.

TABLE II

| Q | T | t (Years) |
|---|---|---|
| 0.00 | 0.000 | (1.0) |
| 0.10 | 0.005 | 2.5 |
| 0.15 | 0.011 | 4.3 |
| 0.20 | 0.019 | 6.6 |
| 0.25 | 0.030 | 9.9 |
| 0.30 | 0.043 | 13.8 |
| 0.35 | 0.058 | 18.2 |
| 0.40 | 0.076 | 23.6 |
| 0.45 | 0.097 | 29.8 |
| 0.50 | 0.120 | 36.6 |
| 0.55 | 0.145 | 43.8 |
| 0.60 | 0.175 | 52.7 |
| 0.65 | 0.212 | 63.7 |
| 0.70 | 0.248 | 74.6 |
| 0.75 | 0.294 | 88.4 |
| 0.80 | 0.348 | 104.3 |
| 0.85 | 0.423 | 126.4 |
| 0.90 | 0.525 | 157.0 |
| 0.95 | 0.699 | 208.7 |
| 1.00 | $\infty$ | $\infty$ |

The tabulated values are plotted in Fig. 6. It can be seen that the theoretical degree of consolidation corresponding to the time when the samples were removed, in 1927, agrees remarkably well with the value of 25 per cent determined by tests.

These results point to the interesting conclusion that of the total amount of excess water which was left in the core during construction, about three-quarters still remained seven years after completion. The fact that considerable excess water was present became strikingly apparent during the sinking of the shaft used for obtaining samples. The reservoir collects water only during floods, and is, therefore, empty most of the time. Nevertheless, about eighteen feet of water had to be pumped out of the shaft each morning, even when the bottom of the shaft was well above the highest recorded reservoir level.

This situation emphasizes the writer's earlier statement to the effect that rapid consolidation of the core is a minor matter as far as stability is concerned. There can be no question as to the stability of the Germantown Dam. Analysis shows that even if the core had remained liquid the design would have been safe. With only 25 per cent consolidation, the core became reasonably stiff, so that the present factor of safety is much more than sufficient. Further consolidation, which will go on for a considerable period of years, will merely make the factor of safety greater and greater. An attempt to obtain more rapid consolidation would have meant a waste of good material and a sacrifice in water-tightness, without any compensating gain whatsoever.

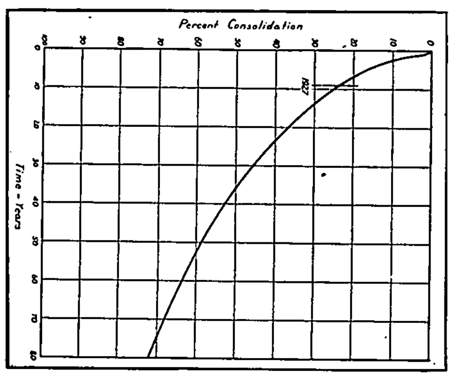


FIG. 6

## SEEPAGE

In a previous paper* the writer outlined a simple method, based on studies made by Dr. Leo Casagrande,† for estimating the seepage through earth embankments. The geometrical simplicity of a hydraulic-fill core permits corresponding simplifications of the former method, which was intended to be a rather general solution.

It was shown that the rate of seepage, $q$, per unit length of a homogeneous embankment resting on an impervious underground could be closely approximated by the equation

$$q = k\ m\ H\ sin\ \alpha$$

* "Hydraulic-Fill Dams." World Power Conference, 1933.

† "Näherungsverfahren zur Ermittlung der Sickerung in geschütteten Dämmen auf Undurchlässiger Sohle." Die Bautechnik, 1934, Heft 15.

The quantities involved are illustrated in Fig. 7. The only troublesome element in the equation is the factor $m$, which expresses the position of the point at which the line of saturation intersects the downstream

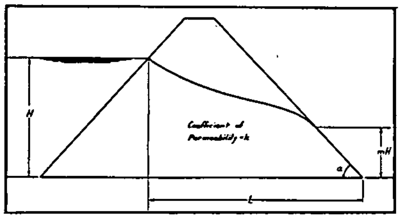

FIG. 7

slope. An implicit relation between $m$ and the other quantities was developed, as follows:

$$\frac{2L}{H}=m \cot \alpha+\frac{\sqrt{1-m^2 \sin^2 \alpha}}{m \sin \alpha}-m \sin \alpha \ln \frac{1+\sqrt{1-m^2 \sin^2 \alpha}}{m\,(1+\cos \alpha)}$$

A set of curves was drawn to enable $m$ to be found for any given case.

In the case of a hydraulic-fill core, the section can be taken as an isosceles triangle. If $H_0$ is the total height, and the depth of water, $H$, is called $nH_0$, the geometry of the figure requires the following relation:

$$\frac{2L}{H}=2 \cot \alpha \left(\frac{2}{n}-1\right)$$

Substituting in the preceding equation —

$$\frac{2}{n}=1+\frac{m}{2}+\frac{\sqrt{1-m^2 \sin^2 \alpha}}{2m \cos \alpha}-\frac{m \sin^2 \alpha}{2\cdot \cos \alpha} \ln \frac{1+\sqrt{1-m^2 \sin^2 \alpha}}{m\,(1+\cos \alpha)}$$

Normally, $n$ would be known and $m$ desired. The expression is still implicit in $m$, so that an indirect solution is required. A plot showing the relation between $n$ and $m$ for various core slopes is given in Fig. 8. When $m$ is found, the seepage per unit length is given by —

$$q=k\,m\,n\,H_0 \sin \alpha$$

Or, to keep the same notation previously used, with cot $\alpha$ denoted by $A$ —

$$q=\frac{k\,m\,n\,H_0}{\sqrt{1+A^2}}$$

As an illustration of the method, let it be required to estimate the seepage per foot of length of the Germantown Dam at its maximum section, 110 feet high, with 95 feet of water in the reservoir. This is not a very good example to use, because the slow rate of consolidation previously determined shows that the seepage must be extremely small. Nevertheless, the simplicity of the method can be demonstrated.

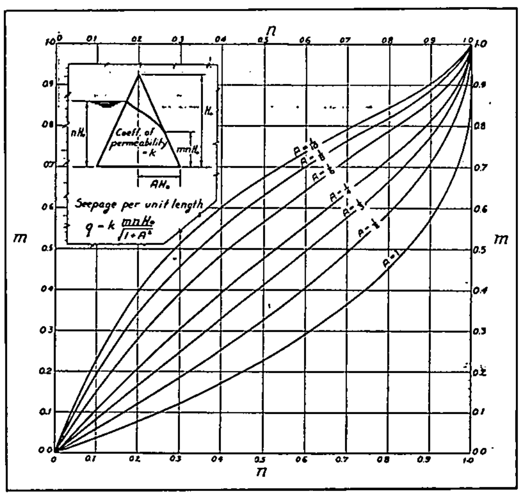


FIG. 8

The factor $n$ is $95 \div 110$, or 0.86. The core slope is 2 on 1, so that $A$ is $\frac{1}{2}$. From the graph, $m$ is found to be 0.66. Therefore —

$$q = k\,\frac{0.66 \times 0.86 \times 110}{\sqrt{1.25}} = 56\,k$$

Since $H_0$ is expressed in feet, $k$ must be in feet per unit time. Using feet per day, $q$ will be in cubic feet per day per foot length. Permeability tests on the core samples gave an average value of $k$ of about 0.00001 centimeters per minute, which is equivalent to about 0.00047 feet per day. Therefore the seepage will be $56 \times 0.00047$, or 0.026 cubic feet per day per foot length of dam.

It should be noted that this method gives only the seepage through the core itself, and does not take care of possible leakage in the underground beneath the dam. In many cases the latter amount will be of first importance. Even when a diaphragm of concrete or sheet piling is used to cut off the underground currents, considerable water may get through by traveling in the underground to the vicinity of the diaphragm, then up and around it in a sort of "short-circuit" through the lower portion of the core. The conditions governing effects of this type are of so variable a nature that little can be done in the way of a general analysis. Each individual case must be carefully studied, with a view to adopting a design suitable to the particular situation.

## Conclusion

The theories herein presented have been developed for the purpose of clarifying and rationalizing, as far as possible, the physical problems involved in hydraulic-fill construction. The simplifications and idealizations used are no more radical in nature and extent than those associated with other types of engineering theory, such as the concepts of frictionless joints in trusses, simple bending of beams, and so on. Obviously, the practical difficulties encountered in construction will frequently necessitate departures from theoretical ideals. Nevertheless, it is the writer's belief that this general method of attack should be very useful as a guide to proper analysis, and should assist materially in the correlation of present and future information on the subject.

# Appendix

## Derivation of Stability Equation

The problem of the stability of the shell of a hydraulic-fill dam can be stated in various ways. For the present analysis it will be stated as follows: given the inner and outer slopes of the shell, its weight per unit volume, and its angle of internal friction; to find the maximum core pressure which the shell can withstand.

In the accompanying diagram, the inner and outer slopes are represented by lines $OD$ and $OE$, making angles $\alpha$ and $\beta$, respectively, with the horizontal. The weight per unit volume of the shell will be called $w$. Its angle of internal friction will be represented by $\phi$. The length in the third dimension will be taken as unity.

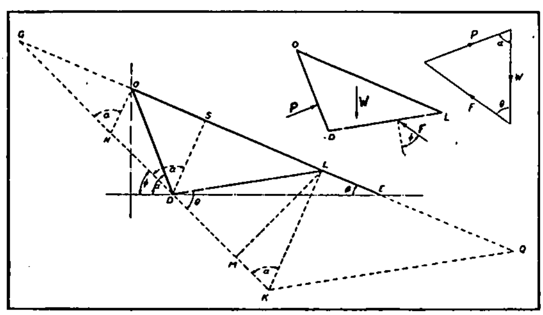

Fig. 9

Through $D$ draw a line making an angle $\phi$ with the horizontal, intersecting $OE$ produced in $G$. Assume that the surface $DL$, making an angle $\theta$ with this line, is the surface along which danger of slipping is a maximum. It is desired to determine the position of $DL$ in terms of known quantities.

The portion of the shell above this surface, triangle $ODL$, is in equilibrium under the action of three forces, as shown in the separate sketch: its own weight, $W$; the core pressure, $P$; and an internal resultant force, $F$, making an angle $\phi$ with the normal to $DL$.

From the force triangle, it can be seen that —

$$P = W \frac{\sin \theta}{\sin (\alpha + \theta)}$$

Since $DL$ is the sliding surface, $P$ must be a minimum for this particular value of $\theta$. Therefore $\frac{dP}{d\theta}=O$. Differentiating the equation for $P$ —

$$\frac{dP}{d\theta}=\frac{\sin\theta}{\sin(\alpha+\theta)}\frac{dW}{d\theta}+W\frac{\sin\alpha}{\sin^2(\alpha+\theta)}=O.$$

or

$$W=(-\frac{dW}{d\theta})\sin\theta\frac{\sin(\alpha+\theta)}{\sin\alpha}$$

The value $(-\frac{dW}{d\theta})$ may be written $\frac{1}{2}w(DL)^2$.

Also, if $KL$ is drawn making an angle $\alpha$ with $GK$, and $ML$ is perpendicular to $GK$, it is evident that —

$$(DL)\sin\theta=ML,$$

and

$$\frac{\sin(\alpha+\theta)}{\sin\alpha}=\frac{DK}{DL}$$

Substituting in the previous equation —

$$W=\frac{1}{2}w(DK)(ML).$$

The right side of this equation is merely the area of triangle $DLK$ multiplied by $w$, whereas $W$ is the area $ODL$ multiplied by $w$. Therefore, triangle $DLK$ = triangle $ODL$. The problem is thus reduced to finding the position of point $K$ such that when $KL$ and $DL$ are drawn, the triangles will be equal in area.*

Draw $ON$ parallel to $KL$, and $KQ$ parallel to $DL$.

In triangles $GDL$ and $GKQ$, $\frac{DK}{DG}=\frac{LQ}{LG}$

In triangles $GNO$ and $GKL$, $\frac{NK}{GK}=\frac{LO}{LG}$

But triangle $ODL$ = triangle $DLK$, so that $LQ=LO$.

Hence

$$\frac{DK}{DG}=\frac{NK}{GK}=\frac{DN+DK}{DG+DK}$$

or

$$\frac{DN}{DK}=\frac{DK}{DG}$$

* This relation is similar to "Rebhann's principle" in the theory of retaining walls.

That is, the length $DK$ is a mean proportional between $DN$ and $DG$; and, since points $N$ and $G$ are known, point $K$ can readily be found by the usual mean proportional construction.

Furthermore, the value of $P$ can be determined as soon as these steps have been made. The first equation for $P$ can be written —

$$P = \tfrac{1}{2} w (DK)(ML) \frac{\sin \theta}{\sin(\alpha + \theta)}$$

From triangle $DKL$ —

$$\frac{\sin \theta}{\sin(\alpha + \theta)} = \frac{KL}{DK}$$

Substituting —

$$P = \tfrac{1}{2} w (KL)(ML)$$

The procedure for a graphical solution may now be summarized as follows:

1. On a scale layout of the section, draw through $D$ a line making an angle $\phi$ with the horizontal, intersecting the outer slope in $G$, and extending in the opposite direction toward the unknown point $K$.
2. Draw $ON$ making an angle $\alpha$ with $DG$.
3. Find by geometrical construction a line of such length as to be a mean proportional between $DN$ and $DG$. Lay off this length below $D$, thus determining point $K$.
4. Through $K$ draw a line parallel to $ON$, intersecting the outer slope in $L$.
5. Then $DL$ is the trace of the surface along which sliding will tend to take place, and the maximum core pressure will be the weight of shell material in a triangle of base $KL$ and altitude $ML$.*

For an analytical solution it will be convenient to eliminate the absolute value of $P$ and reduce everything to abstract ratios. If the unit weight of the core material is $Rw$, the pressure $P$ can be written in the usual hydrostatic form:

$$P = \frac{Rw}{2} (DO)^2 \sin \alpha$$

Also $$P = \frac{w}{2} (KL)(ML) = \frac{w}{2} (KL)^2 \sin \alpha$$

Therefore $$\sqrt{R} = \frac{KL}{DO}$$

Draw $DS$ parallel to $KL$ and $NO$.

* This construction is similar to that of Poncelet for the earth pressure on a retaining wall.

From the similar triangles $GKL$ and $GDS$ —

$$\frac{KL}{DS}=\frac{KG}{DG}=1+\frac{DK}{DG}$$

From triangle $DOS$ —

$$\frac{DO}{DS}=\frac{\sin(\alpha-\beta+\phi)}{\sin(\alpha-\beta)}$$

Whence —

$$\sqrt{R}=\frac{KL}{DO}=\frac{\sin(\alpha-\beta)}{\sin(\alpha-\beta+\phi)}\left[1+\frac{DK}{DG}\right]$$

From the mean proportional relation —

$$DK=\sqrt{(DN)\,(DG)}\text{ or }\frac{DK}{DG}=\sqrt{\frac{DN}{DG}}$$

In triangle $DNO$, $\dfrac{DN}{DO}=\dfrac{\sin\phi}{\sin\alpha}$

In triangle $DGO$, $\dfrac{DG}{DO}=\dfrac{\sin(\alpha-\beta)}{\sin(\phi-\beta)}$

Whence — $\dfrac{DN}{DG}=\dfrac{\sin\phi\,\sin(\phi-\beta)}{\sin\alpha\,\sin(\alpha-\beta)}$

Therefore — $\sqrt{R}=\dfrac{\sin(\alpha-\beta)}{\sin(\alpha-\beta+\phi)}\left[1+\sqrt{\dfrac{\sin\phi\,\sin(\phi-\beta)}{\sin\alpha\,\sin(\alpha-\beta)}}\right].$

This equation gives the required relation between the variables. A somewhat more convenient form can be obtained by introducing the notation —

$$A=\cot\alpha$$
$$B=\cot\phi$$
$$C=\cot\beta$$

The trigonometric functions can then be expressed in terms of these coefficients. The substitutions, expansions and reductions are straightforward algebraic operations, and need not be described in detail. The solution assumes the form —

$$\sqrt{R}=\frac{(C-A)\sqrt{1+B^2}+\sqrt{C-A}\,\sqrt{C-B}\cdot\sqrt{1+A^2}}{(1+C^2)-(C-A)\,(C-B)}$$

## Discussion

DR. ARTHUR CASAGRANDE,* MEMBER

Professor Gilboy's paper contains the results of many years of experimental and theoretical researches which he conducted with great perseverance in order to build up a "Mechanics of Hydraulic-Fill Dams" based on the principles of modern Soil Mechanics which were established by Professor Terzaghi. Fortunately for the designers of earth dams, we do not find in this paper any vague considerations which are of little or no use to the engineer, but very tangible results of lasting value.

Perhaps the most outstanding achievement is the realization that the stability of such a structure is mainly dependent on the design of the outer sections. The proposed methods for analyzing the stability are so simple and logical that they will undoubtedly become the backbone of future hydraulic-fill dam design. A very convincing proof for the accuracy of the basic assumption of a plane shearing surface is given by the ingenious model tests. The fundamental influence of the shape of the core on the stability of the structure is a rather unexpected and most important finding.

It is a rather pleasant aspect for the designer to know that the composition of the core is not of such vital importance to the stability of the dam as has been so far assumed, and it leaves him a much wider margin for the types of materials which he may consider suitable for the sluicing of the core. On the other hand, the question arises whether the construction of this type of earth dam is to be limited to such cases where an abundant supply of coarse-grained soils is available for the construction of the outer sections, or whether a satisfactory hydraulic-fill dam can also be built with very limited quantities of coarse materials. In the opinion of the writer the use of fine-grained soils in the outer sections should be possible if, by means of sufficient drainage facilities, *e.g.*, horizontal layers of coarse-grained material, properly spaced, the fine-grained layers are allowed to consolidate almost as rapidly as the load during construction is increased, and if, in the design, a conservative angle of internal friction is used for the fine-grained shell material. In the reconstruction of the Alexander Dam in Hawaii, Mr. J. B. Cox† has successfully applied a drainage system in the outer sections of the

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* Graduate School of Engineering, Harvard University.

† J. B. Cox: "Beach Drainage Safeguards, Alexander Dam." Engineering News Record, October 20, 1932.

dam, the bulk consisting of a rather impervious, fine-grained volcanic soil which, without such precautionary measures, could hardly be considered safe for such a purpose.

While in the majority of cases the rate of consolidation of the core does not influence the stability of the structure to such an extent that it should be considered in the design, there are instances where such an analysis will be very helpful. Professor Giboy's solution of this intricate problem permits a relatively simple application to actual conditions. His interesting experimental investigations on the Germantown Dam of the Miami Conservancy District corresponded very well with the theoretical analysis.

Professor Gilboy suggests a very practical method for determining the seepage through the core which is based on more reliable assumptions than has heretofore been customary. This method should also become very useful for the analysis of seepage through other types of earth dams, dikes, etc.

Professor Gilboy is to be congratulated for having made the most outstanding contribution to the subject of hydraulic-fill dam design. He has, for the first time, clearly defined the purpose of the various sections of such dams and devised scientific and yet very practical methods for the design permitting a greater degree of safety combined with more economical designs in which the available materials are utilized to the best advantage.

# THE APPLICATION OF THEORIES OF ELASTICITY AND PLASTICITY TO FOUNDATION PROBLEMS

By LEO JÜRGENSON, Sc.D.*

(Presented at a meeting of the Designers Section of the Boston Society of Civil Engineers held on May 9, 1934)

THIS paper deals with a method for analyzing the stability of foundations against shearing failure by computing the maximum shearing stresses in the underground and comparing these stresses with the shearing strength of the material.

The following notation is used for stresses:

$p$ = external pressure.
$n_x$ = normal stress on vertical plane.
$n_z$ = normal stress on horizontal plane.
$n_1, n_2$ = principal normal stresses.
$s_{zx}$ = shearing stress in $zx$ plane.
$s$ = principal shearing stress.
$c$ = shearing strength.
$w$ = weight per unit volume.

The term "plastic state" is defined in Appendix 1, (A). In order to maintain continuity of the main text, all mathematical derivations and explanations are given in the Appendix.

The method of applying the Theory of Elasticity to foundation problems is analogous to the use of the same theory in structural design. In designing a truss we first find the stresses in a given member and then compare them with the strength. The same procedure can be followed in many problems of foundation engineering. We first find the stresses that a given system of external forces produces in the ground, and second, determine the strength of the material to see if it can resist the imposed stresses without causing excessive deformations.

Deformations, in general, can be resolved into two main classes: (1) volume change, and (2) distortion due to shear.

Volume change is caused by normal stresses tending to compress the clay mass. The voids of clay are filled with water which has to

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escape before the material can compress, thus introducing the factor of time into the process. The magnitude and the time rate of deformations due to volume change can be analyzed by the Theory of Consolidation. [(1), (7), (8), (9).]* The present discussion will be limited to the question of deformations caused by shear.

Distortion or change in shape is caused by shearing stresses. Depending upon the magnitude of the shearing stresses, and upon the shearing strength of the material, it may be only a slight deformation proceeding very slowly, or it may be a rapid slide resulting in considerable deformation.

The magnitude and the rate of a gradual distortion are very difficult to analyze. It is much easier to deal with the question of ultimate

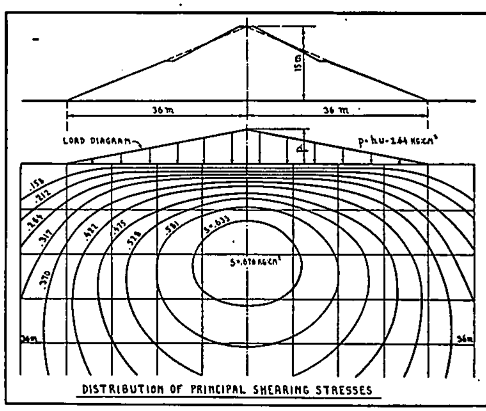



FIG. 1

resistance, *i.e.*, the resistance to failure by sliding. We can do this by comparing the stresses produced in the ground by given external loads with the ultimate shearing resistance of the soil. The analysis thus resolves itself into two main parts: first, determination of the shearing stresses; second, determination of the shearing strength of the underground materials.

The first part of the problem is a theoretical computation involving the theory of elasticity, and its result gives us a picture of the distribution of shearing stresses under the given system of external loads. This information can be conveniently plotted in cross sections of the type shown in Fig. 1.

* The numbers in brackets refer to Bibliography at end of report.

The second step is more empirical. It involves taking samples of the material, testing them and interpreting the results. We thus get a complete picture of underground conditions for a given case. A comparison of the data so obtained serves as the basis for deducing the answer to a given problem. Whenever possible, such an analysis should be compared with a similar analysis of an existing structure on the same material to arrive at a truer interpretation of the results of the laboratory tests.

To demonstrate the use of this method by an example, analyze the question of stability against shearing failure for the case shown in Fig. 1. This represents an earth fill dam 15 m. high and 72 m. wide at the base. Taking 1.76 g./cm.$^3$ (110 pounds per cubic foot) as the average weight of the fill, and simplifying the outlines of the structure as indicated, the pressure distribution on the base line can be represented by a triangle of height $p = 1500 \times .00176 = 2.64$ kg./cm.$^2$. The distribution of principal shearing stresses which such a loading produces in the underground is indicated in the figure. The stresses have been computed by the Theory of Elasticity, assuming the underground to be isotropic and homogeneous and of infinite extent. The lines on the diagram connect points of equal stress intensity. The shearing stress is highest at a point under the center of the dam at a depth of 18 m. below the ground surface. In the given case the maximum intensity is 0.676 kg./sq. cm.

This is the first quantity to compare with the shearing strength of the material. If this stress is lower than the strength, the answer is evident; if higher, it does not necessarily follow that danger of a slide exists. If in our case the underground has a uniform strength of $c = .63$ kg./cm.$^2$, we see that only inside the $s = .63$ line on the stress diagram is the strength exceeded by the stresses. This danger zone is confined and is surrounded on all sides by material that still has a reserve of resisting capacity. The material in the plastic zone will yield somewhat and will transmit to the adjoining material that part of the load which it cannot resist itself.

The theoretical stress diagram is then no longer correct, as the plastic region will actually be larger than the area inside the .63 shear line. If the external loads are further increased, the plastic zone is again extended until failure occurs when the ultimate resistance of the underground is exceeded.

We thus see that in addition to finding the maximum shearing stress we also have to consider where it occurs and to what possible consequences it may lead. Only in the case where a progressive failure is possible does the mere exceeding of maximum shearing strength lead to a slide.

When computing the stresses it is of advantage to consider separately the stresses due to external loads and the stresses due to weight of soil. This procedure materially facilitates the computations. Once we have calculated the stresses due to a given system of external loads, we can apply the results to any similar problem by merely changing the linear scale of the stress distribution diagram. Thus the diagram shown on Fig. 1 can be used for any long loaded area with triangular loading resting on uniform material. The stresses are proportional to the intensity $p$, and the linear dimensions are proportional to the width of the base. If the dam were twice as high, the maximum shear would be 1.35 kg./cm.$^2$, and would occur at the same point. In other words, the maximum intensity is always 0.256 $p$ and occurs at a depth equal to 0.2502 times the total width of base. (See Fig. 9.)

The stresses due to the weight of soil can be added to stresses caused by external loads, or they can be taken into account when considering the strength of the material. Materials of low permeability, such as fat clays, derive their total strength from pressures to which they have been consolidated before we apply our external loads. Not until the clay begins to adjust itself to the newly imposed pressures and begins to lose excess water do the imposed stresses begin to contribute to its strength. The strength of the material is therefor increasing with time at a rate which can be analyzed by the Theory of Consolidation. In impervious materials this time lag is so great that we can rely only on the shearing strength of the material in its natural state. (Appendix I, (A).) In case of impervious materials the effect of stresses due to the weight of soil itself is therefore automatically included in the strength of the material.

## Determination of Stresses

Formulæ based on the Theory of Elasticity for the cases which occur most frequently are given in Figs. 12 and 13. [(2), (3), (4), (5), (1), page 222.] We shall not discuss here the question of how far this theory is applicable to a material like soil. The theory presupposes an elastic, homogeneous and isotropic material, and for most cases it is the only theory that is available for our purpose. Only for a few cases have solutions been made by Rheology (the Theory of Plastic Flow). Application of one such solution is demonstrated in Example 4.

In employing the Theory of Elasticity attention should be paid to cases where there are sharp variations in the structure of the underground. The presence of a very soft layer in an otherwise homogeneous ground changes the stress distribution materially. To illustrate this, let us consider the case of a long strip of width $2b$ subjected to a uniform pressure $p$. The distribution of stresses in a homogeneous material is shown in Fig. 8. The maximum shearing stress occurs on the semicircle passing through the sides of the footing. Its magnitude is $p:\pi$. If there is a surface having no shearing strength at a certain depth $h$, the principal shearing stress is $s=p \tanh \frac{\pi b}{2h}$. [(5), page 539.] This gives us a maximum $s=p$, if $h<=b$. (Appendix I, (F).) The case is more favorable if, instead of a layer having no shearing strength, there is a rigid surface at a moderate depth below the foundation. This would be the case if a deposit of clay rests on rock. The shearing stress at the rock surface is $s=\frac{1}{2}\, p\, [\operatorname{sech} \frac{\pi}{2} \frac{x+b}{h} - \operatorname{sech} \frac{\pi}{2} \frac{x-b}{h}]$. [(5), page 541.] For $h=\frac{1}{2}\, b$ this gives a maximum $s=\frac{1}{2}\, p$. It is of interest to note that this is higher than the shearing stress in a homogeneous underground where the maximum is $p\div\pi$. The presence of rock increases the shearing stress by about 60 per cent. It may be well to repeat at this point that an excessive shearing stress may not yet lead to a slide if the plastic zone is deep in the ground and surrounded by other strata that still have a reserve of strength. In our case, if the material is homogeneous and the foundation rigid a slide would occur if $p=(\pi+2)\, c$. [(6).] If there is a rigid rough surface at depth $h=\frac{1}{4}\, b$, the critical pressure is $p=4c$ (Appendix I, (C)) which is only 78 per cent of the ultimate bearing capacity of a homogeneous underground. Summarizing, we thus get the following comparison:

| | Plastic State begins | Ultimate Resistance |
|---|---|---|
| Homogeneous ground . . . . . . | $p=3.14\, c$ | $p=5.14\, c$ |
| Rock at depth $h=0.25b$ . . . . . | $p=2c$ | $p=4c$ |

As is seen from the above comparison, the elastic stresses and the danger of failure by sliding are both increased by the presence of a rock surface. As was pointed out previously, we are here not considering settlements due to direct compressibility of the clay.

Tables and diagrams given in Appendix II show the distribution of stresses for the following types of loading: uniformly loaded long footing, uniformly loaded circular footing, long footing with triangular loading and a terrace loading. These results refer to homogeneous ground and can be applied to any given case by merely adjusting the linear scale. If two or more systems of external loads are to be superimposed, the stress components $n_z$, $n_x$ and $s_{xz}$ due to each separate system must be added and the resultant principal shearing stress computed by the familiar formula of Elasticity $s=\frac{1}{2}\sqrt{(n_z-n_x)^2+4s^2_{xz}}$. [(18), pages 15–20; (14), page 46.] As the normal stresses $n_z$ and $n_x$ are always compressive, attention has to be paid to the direction of shearing stresses only when adding the components. The stresses need be determined for critical points only, and for a rough approximation a scalar sum of principal shearing stresses is often sufficient. (See Example 3.) Shearing stresses shown in the diagrams are the principal shearing stresses, *i.e.*, the maximum shearing stress at the given point. The tables contain all stress components, the principal stresses and their directions. They can therefore be used in other problems besides those of stability. Attention to the boundary conditions has to be paid, also, when figuring the normal stresses as in the problems of settlements due to consolidation. (Appendix I, (F).)

The author has used the solution for circular areas illustrated in Fig. 10 in cases where the footing is square. The exact solution for the latter case is difficult, and a circle of the same area appears to be a permissible approximation for practical purposes. The formulæ for stresses are omitted as they contain functions of derivatives of the solid angle subtended by the loaded circle at the point under consideration, and are of interest to the mathematician only.

Using the solutions given for a single load and for a long strip as a basis, stresses can be computed for other systems or combinations of external loads by integration. Often an approximate integration has to be used as rigorous solutions become too involved.

Computations for stresses for different and more complicated boundary conditions have to be carried out to suit the particular case. For examples and for methods of arriving at solutions the reader is referred to publications of Carothers and Michell. [(10), page 156; (5).]

## DETERMINATION OF THE SHEARING STRENGTH

For a pervious material the shearing strength is $s=n\ tan\ \phi$, where $n$ is the normal pressure and $\phi$ the angle of internal friction. This formula cannot be used indiscriminately for materials of low permeability, such as clays. Here it applies only if the pressure has been acting long enough to allow escape of the excess water and complete internal adjustment to the given external pressure. The length of time required for this adjustment depends upon the permeability of the material and upon the drainage conditions. Conversely, if the pressure on consolidated clay is released, the shearing strength does not drop to zero as is the case with cohesionless sands. The shearing strength of clay is, therefore, also dependent upon the past history of the deposit, and has to be determined by tests in each particular case. Three types of tests which can be used for this purpose are briefly described below.

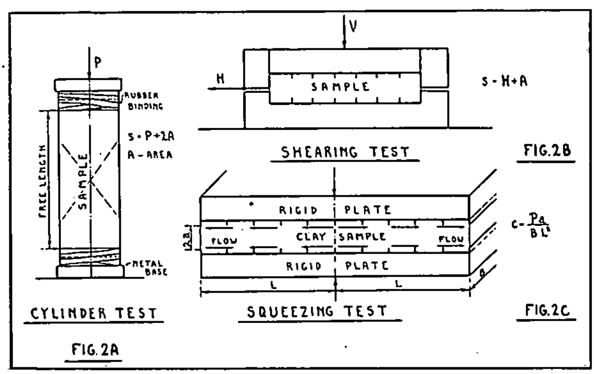



FIG. 2

*Direct Shearing Test*

In this test the sample is inserted between two gratings and the force $H$ is measured which produces a shearing failure of the material. (See Fig. 2B.) In order to prevent the upper grating from rising and losing its grip on the sample, it is always necessary to perform the test under pressure of a vertical force $V$ applied on the upper grating as shown in the figure. As we want to find the shearing strength of the clay in its natural state, care has to be taken to prevent further consolidation of material during the test. The force $V$ must therefore not be excessive, the gratings must be impervious, and the test must be run fairly rapidly. In tests represented in Fig. 7 the duration was three minutes. The shearing gratings were of bronze with hard rubber spacers and with teeth of 0.2 mm. spring bronze 2.5 mm. high, spaced 9 mm. apart. All joints were waterproofed. Possibility for drainage exists, therefore, only along the edges of the sample.

### *Cylinder Test*

A cylinder of clay, cut with a fine wire saw, is subjected to compression to failure as shown in Fig. 2A. Care should be taken in eliminating the effects of irregularities at the ends of the sample. If the end plates are very slippery, the sample will fail by bulging sharply at one end. In soft clays a paper or emery cloth disc is usually sufficient to prevent this. In most cases, especially in case of very stiff clays, it is necessary to bind the ends with a strip of thin rubber. The failure plane is thus forced into the middle of the sample and away from local stress concentrations at the ends. The distribution of stresses on the failure plane is, therefore, at least as uniform as the material itself. The intensity of the maximum shearing stress is one-half of the principal stress. In a relatively impervious material (condition of plasticity $2c=\sqrt{4s^2+(n_z-n_x)^2}$, Appendix I, (A)), the planes of maximum shear should also be the failure planes. Their directions are at 45° to the principal planes.

In a pervious material (condition of plasticity $\frac{1}{2}\sqrt{(n_z-n_x)^2+4s^2}=\frac{1}{2}(n_z+n_x)\sin\phi$, Appendix I, (A)), the rupture planes are at $45°+\frac{1}{2}\phi$ to the plane of maximum principal stress. Such planes can easily be observed and measured on the surface of the sample. In the case of cohesionless sands the cylinder test has to be performed under lateral pressure. The surface of the cylinder is covered with a skin of thin rubber, and the lateral pressure is applied by means of compressed air. The angle of friction determined from the inclination of failure planes agrees in this case with the value computed from the stresses at failure.

As clays are relatively impervious materials, the practice of estimating their angles of internal friction from cracks on common compression test cylinders would appear to the author to be erroneous.

### *Squeezing Test*

This test, which can be used for relatively impervious clays only was recently devised by the author and is described more in detail in a separate paper. A thin sample of material is placed in a rigid box open on two opposite sides, and pressure is applied through a rigid top plate so that the material is squeezed out on the open sides. (See

Fig. 2C.) The surfaces of the top and bottom plates in contact with clay must be so constructed as to mobilize the full shearing resistance of the material and prevent drainage. The writer used 75×92 mm. steel plates with teeth of 0.2 mm. spring bronze spaced 9 mm. apart and projecting 2.5 mm. into the clay. The thickness of the samples was 10 to 25 mm. The sides of the box were lined with glass and soft rubber to avoid jamming of the top plate and to prevent escape of material. All parts were waterproofed and the tests were run fast enough to prevent consolidation of the material during the test. This latter factor is very important because in this test the clay is under a high normal stress. The formula given in Fig. 2C is derived in Appendix I, (B).

### *Comparison of Testing Methods*

In the shearing test the sample can fail along the single surface which offers the least resistance. Therefore this test is very sensitive to any local defect, non-uniformity of material, or stress concentration. The same applies also to the cylinder test. However, the failure plane in the cylinder test is far away from the ends where the forces are applied, and the stress distribution is more uniform.

In the squeezing test one single sliding plane cannot lead to failure as the material must shear along a number of planes. Since natural materials are never quite uniform, this is a desirable feature, as the test reveals the average strength of the material and not its strength along the weakest plane in a small-sized sample.

The cylinder test is the simplest to make and requires no complicated apparatus.

Fig. 7 shows a summary of test results performed on a sample of Boston blue clay taken with a 5-inch diameter sampler from a depth of 54 feet. The variations in results are due to natural non-uniformity of the deposit and disturbance in taking the sample.

A more detailed treatment of the subject of shearing strength and testing methods is given in a separate paper.

## Comparison with the Ultimate Resistance to Penetration

The stresses computed by the Theory of Elasticity do not hold beyond the point where the maximum shearing stress reaches the shearing resistance of the material. If the pressure be increased beyond this limit, a plastic zone will develop in the ground, as was discussed earlier. The stresses at this semi-plastic stage are rather complicated. Solutions are more easily obtained for the fully plastic state when the ground has reached its ultimate resistance and failure occurs. It may be clearer if we illustrate this by an example, considering the case of a uniformly loaded isolated footing. Although the commonest shape is rectangular, we will here assume it to be equivalent to a circle of the same area, as the factor of rotational symmetry simplifies the problem.

As is seen from Fig. 10, the plastic state begins under the edge when $p=\pi c$. If the pressure be increased, the plastic zone will yield somewhat and the pressure on the base will change from a uniform distribution to one gradually increasing toward the center. At the critical point, when a slide is incipient, the pressure will be $(\pi+2.52)\ c$ at the center, and $(\pi+2)\ c$ at the edge. [(13), page 250.] The ultimate mean pressure is $p_m=5.64\ c$. We thus see that the failure occurs when the load is increased about 80 per cent beyond the load at which the plastic state first appeared, provided the foundation is rigid. If this condition were not fulfilled, a partial slide could have taken place much earlier, because in this case the pressure distribution is in its effect not equivalent to the condition of rigidity. Only in exceptional cases, as in the case treated in Example 4, is the stress diagram such that its effect is equivalent to rigidity of the loaded area.

We thus see that in addition to the possibility of a progressive failure there are other limitations which must be considered before the formulæ for ultimate resistance to penetration can be applied to foundation problems. As it is easy to overlook the limitations and assumptions on which these formulæ were derived, it would appear to the author that in general use the rule of keeping the maximum shearing stresses below the strength of material would have less chance of being misapplied.

Comparison with the Swedish method of circular slides is illustrated in examples which follow.

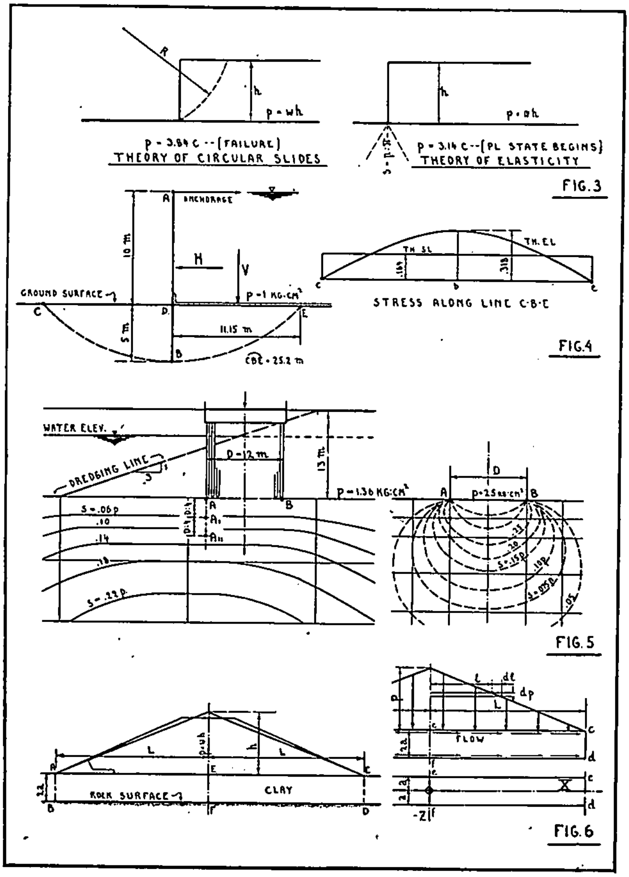
*Example 1.* — Consider the stability of a 90° slope of height $h$. (Fig. 3.) According to the Theory of Elasticity the maximum shearing stress occurs at the toe of the slope and is $s=p:\pi=.318\ p$. (Formula 1, Fig. 13.) According to the Swedish theory of circular slides the failure would occur at $c=.26\ p$ where $c$=shearing strength of material. [(18), (16).] Thus we have for comparison:

| | | |
|---|---|---|
| Theory of circular slides | $p=3.84c$ | (failure occurs) |
| Theory of Elasticity | $p=3.14c$ | (plastic state begins at toe) |

The Swedish method assumes a uniform unit resistance all along the circular failure plane. In the case of materials which lose most of their shearing strength when disturbed (*e.g.*, Laurentian clays) [(9)]

a slide could begin when the plastic limit is reached and proceed as a progressive failure.

When the slope is flatter the danger zone recedes from the toe of the slope and goes deeper into the ground. The depth of this zone is proportional to the horizontal projection of the length of the slope.




Figs. 3, 4, 5 and 6

(See Fig. 11.) It is of interest to note that the most dangerous sliding plane computed by the Swedish method also goes deeper into the ground when the slope is flatter.

*Example 2.* — Consider the stability of the cofferdam wall A–B, shown in Fig. 4. For comparison of the two methods consider only the shearing stresses caused by the vertical component of the water pressure. For simplicity assume an impervious blanket to cover the sea bottom in order to eliminate the hydraulic computations. Taking moments around point A, we get $M = \frac{1}{2}\ 1115^2 \times 1.0 = s \times 1500 \times 2520$. Hence $s = 0.164$ kg./cm.$^2$

According to the Theory of Elasticity the maximum shearing stress is $s = p : \pi = .318$ kg./cm.$^2$ and occurs on the vertical plane passing through point D. The diagram in Fig. 4 shows the distribution of stresses along the arc CBE according to the two principles. In computing the stresses according to the Theory of Elasticity the presence of piles DB in the ground has been disregarded. The plot shows the scalar magnitudes of the principal shearing stresses. At the point of highest intensity the principal shear acts in a horizontal plane, and therefore the plotted stress at this point is true vectorially.

If the material adjoining the sheet piles at DB had been reduced to the plastic state, and could find a way to escape to the surface, the movement would continue progressively. Under such circumstances the condition $c = p \div \pi$ would become the criterion of safety.

As is seen from the above examples, each method would give us a different value for the apparent shearing strength of the clay if we were analyzing an actual slide, namely, $c = .26\ p$ and $c = .318\ p$ in the first example. If the results of the above analyses are applied in a similar way on another project, using in each case the value of $c$ deducted by the corresponding method, both methods would lead to equivalent answers.

If the results of laboratory tests are applied indiscriminately, the final answers by the two methods would differ considerably. It would therefore appear to the author that, if a design is based on results of laboratory tests and not upon results obtained from analysis of an actual slide, the method of computing the maximum shearing stresses by the Theory of Elasticity might in some cases be preferable.

*Example 3.* — Consider the question of stability of the foundation of a silo to be built as shown in Fig. 5. To take into account the combined effect of the silo load and the effect of dredging we will consider them separately and add the resultant shearing stresses. Allowing for the hydrostatic uplift of the submerged material, the pressure on the base line is 1.36 kg.: cm.$^2$ Principal shearing stresses produced by such a terrace loading in the underground computed by formulæ given in Fig. 12 and Fig. 11 are shown on the diagram by full lines. The shearing stresses due to a uniform pressure of 2.5 kg./cm.$^2$ applied on circular area at A–B are shown by dotted lines. If we make for the first approximation a scalar addition of these stresses, we get the following figures (in kg./cm.$^2$):

| Point | A | B | AI | AII |
|---|---|---|---|---|
| $s$ due to silo | .80 | .80 | .73 | .50 |
| $s$ due to embankment | 0 | 0 | .08 | .17 |
| Sum | .80 | .80 | .81 | .67 |
| Overburden $p = wh$ | .68 | 1.19 | .935 | 1.19 |
| $c = .25 + p \tan 32°$ | .672 | .995 | .835 | .995 |
| $c_1 = p \tan 32°$ | .425 | .745 | .585 | .745 |

In the actual case the underground consisted of a relatively impervious mixture of fine sand and clay. Shearing tests showed a strength of .25 kg./cm.$^2$ and an angle of friction of 32°. Mere remoulding without addition of water did not materially weaken the material. When flooded and allowed to swell, the material rapidly lost cohesion and behaved like a sand. The shearing strength of the material due to its own overburden is shown in the above table. The last line gives the shearing resistance the material would have after swelling if all cohesion were lost. Remembering the unfavorable assumptions made in arriving at these stresses, one may conclude that there is no danger of instability. However, the cutting of the canal and the subsequent swelling of the clay would cause a considerable rearrangement of stresses around the waterfront piles. If the method of construction were not altered, this might cause a slight differential settlement tending to tip the high structure.

*Example 4.* — Consider an embankment resting on a clay stratum underlaid by rock as shown in Fig. 6. To eliminate the question of the strength of the structure itself, assume it to be solid and hard. Due to presence of rock the stresses shown in Fig. 9 no longer hold. As is shown in Appendix I (E), the principal shearing stress at the rock surface now is —

$$s = \frac{4a}{L}\,\frac{p}{\pi}\left[2 \text{ arc tan } e^{\frac{\pi}{2}\frac{x}{2a}} - \text{arc tan } e^{\frac{\pi}{2}\frac{x+L}{2a}} - \text{arc tan } e^{\frac{\pi}{2}\frac{x-L}{2a}}\right].$$

If the depth to the rock surface, $2a$, is $\frac{1}{2}$ $L$, this gives us a maximum $s = .318\, p$ at $x = .625\, L$; hence the plastic state at rock surface appears when $p = 3.14\, c$. If $2a = \frac{1}{4}\, L$, $s_{max.} = .22\, p$, at $x = .67\, L$; the plastic zone begins at $p = 4.55\, c$. To analyze the question of the ultimate resistance we will not consider the fully plastic case at the point of failure of the dam by sinking. The plastic material would then flow toward the toes of the embankment and form the familiar "mud waves." Applying a method developed by Hencky for the flow of plastic materials under boundary conditions that, with regard to the stress conditions, are equivalent to those in our problem, the stresses can in this case be computed by a very simple formula.

As shown in Appendix I (D), the shearing stress along the boundaries $ac$ and $bd$ (Fig. 6) is constant and equal to $s = pa/L$. If $c$ is the shearing resistance of the clay, the capacity of the underground would therefore be $p = cL/a$. If, for example, $L = 36$ m., the depth of clay deposit $2a = 9$ m. and the shearing strength $c = 0.4$ kg./cm.², we will get for the critical pressure $p = \frac{cL}{a} = \frac{.4+36}{4.5} = 2.88$ kg./cm. This pressure corresponds to a height of dam of 16.3 m.

It is of interest to note for comparison that the plastic state at rock surface appeared, first, at $x = .67\ L = 25$ m. when $p = 4.55\ c = 1.82$ kg./cm.², *i.e.*, at 63 per cent of the ultimate.

It might be of interest to discuss at this point the measures a designer could use in case the pressure is excessive. As is seen from the above formula, the shearing stresses increase with ($a$) and decrease with $L$. They could be reduced by increasing the width of the dam or by some means of strengthening the underground, such as a partial replacement of clay by a more resistive material. Another possibility would be to build up the height in stages, allowing the material to consolidate and thus increase its shearing resistance between successive stages.

The method of circular sliding surfaces would not be directly applicable to this case, as the mode of failure is different.

*Example 5.* — Consider what cross section of a long storage pile resting on soft clay gives the best storage for a given width if the shearing strength of the clay is not to be exceeded. Call the width $= 2b$, the height of pile $= h$, and the strength of clay $= c$.

For a triangular section we get, according to Fig. 9 —

$$p = wh = \frac{1}{.256}\, c \quad \therefore\ h = 3.91\,\frac{c}{w} \text{ and storage area } A = \tfrac{1}{2}\, h\ 2b = 3.91b\,\frac{c}{w}.$$

For a rectangular section (Fig. 8A) we get —

$$wh = \pi\, c \qquad \therefore\ h = \pi\,\frac{c}{w} \text{ and } A_2 = 2bh = 2\,\pi b\,\frac{c}{w}.$$

Thus a pile of uniform height gives a 61 per cent greater storage capacity. If it is desirable to avoid high stresses near the surface at the edges of the pile, a trapezoidal form could be used. (Last formula, Fig. 13.)

## Appendix I

Plastic state as used in this paper refers to the state where the stresses have reached the shearing strength of the material. Material in the plastic state can deform under a constant shearing stress. In the elastic state the stress must increase with increase of strain.

**(A)** The term "impervious" refers to material that does not change its water content during the test. The material is consequently unable to adjust its internal structure to the rapidly imposed stresses. The pressures are therefore not transmitted through the solid grains but are carried by water trapped between grains and do not contribute to the strength of material. In two dimensional cases the condition of plasticity for such a material is $\sqrt{(n_x-n_z)^2+4s_{xz}^2}=2c=\text{const.}$ (Coulomb's Theory of Rupture.) [(14), pages 60, 184.] The strength of the material depends only upon the maximum shearing stress, *i.e.*, upon the difference between the principal stresses, and is independent of intensity of normal stresses. In terms of principal stresses the above relation is $n_1-n_2=2s=2c=\text{const.}$ For a pervious material, such as coarse sand, the condition of plasticity is $\frac{1}{2}\sqrt{(n_x-n_z)^2+4s_{xz}^2}=\frac{1}{2}(n_z+n_x)\sin\phi$. (Mohr's Rupture Theory.) [(12), (11), (14), pages 45 and 61.] The strength here depends upon the ratio of the two principal stresses. Expressed in terms of principal stresses the law is $\frac{n_1}{n_2}=\frac{1+\sin\phi}{1-\sin\phi}$. Both materials have internal friction, but in an impervious material it cannot develop and manifest itself if the loads are applied rapidly. The material has only the strength produced by pressures to which it had been consolidated previously.

As no material is absolutely impervious, it would follow that, allowing for the time factor, both materials are alike. Investigations, the results of which are at present being assembled for publication, have confirmed this even for very fat clays. Allowing no drainage, the clay followed Coulomb's Law even at pressures carried up to 16 atmospheres. If drainage is allowed, the same clay follows Mohr's Law, revealing the same angle of internal friction as a coarse mica powder.

In nature the deposits are relatively thick and the drainage so slow that it must be neglected in stability computations.

**(B)** If no consolidation is allowed the condition of plasticity of clay in case of plane strain is $2c=\sqrt{4s_{xz}^2+(s_z-s_x)^2}$. For the case of two dimensional plastic flow of such a material, the mathematical analysis has been solved by Hencky. [(13), (14), page 221.] The solution given for parallel boundaries can be applied to the case when a plastic material is squeezed between two rigid plates. [(15).] The author has used this solution as the theoretical basis for a test for determination of the shearing strength of clay. For brevity, the test is referred to as the Squeezing Test.

Taking the co-ordinate axis as shown in Fig. 6, the formulæ for the stresses are —

$$n_x = \frac{c\,(L-x)}{a} \pm 2c\sqrt{1-\frac{z^2}{a^2}} + const.$$

$$n_z = \frac{c\,(L-x)}{a} + const.$$

$$s_{xz} = -c\,\frac{t}{a}.$$

At the boundaries $z=a$ and $z=-a$ —

$$n_z = n_x = \frac{c}{a}\,(L-x) + const. \quad s_{xz} = -c.$$

Integrating the normal stress $n_z$ over the area of the plate from $-L$ to $+L$ and from zero to $B$ in $y$ direction, we get the formula —

$$P = \frac{1}{a}\,cL^2B;\ c = \frac{Pa}{BL^2}$$

**(C)** Making the same assumption as Prandtl [(6)] that the foundation is rigid, we can say that the pressure distribution on the base will adjust itself so as to develop the maximum resistance to penetration. Before failure can occur, the pressure distribution will therefore change to a triangular form, with maximum intensity at the center. For the latter case we can again apply Hencky's solution given above in (B). The vertical stress is $n_z = c\,\frac{(L-x)}{a}$. The total force on the bottom of foundation is $P = 2\int_0^L n_z dx = \frac{cL^2}{a}$. Using the notation of the text $P = \frac{2cL^2}{h}$. Taking $h = .25b$ and $P = 2b\ p$, as before, and substituting, we get the formula given in the text, $p = 4c$. This formula applies only if $h$ is small relative to $b$.

**(D)** The solution given in (B) refers to a plastic material squeezed between two rigid rough plates. The resultant vertical stresses are uniformly increasing toward the center, giving a triangular distribution diagram. Now, we can say, if we apply a triangular loading directly to the material, the conditions will be equivalent to the above case and the stresses in the material will be the same. We can therefore apply Hencky's solution to our problem if we assume the pressures on the base of the dam to be proportional to the height, and simplify the cross section of the dam to a triangle of equal area. The mean length of the rigid boundary can be taken equal to the width of base ($2L$). This neglects the resistance the material encounters beyond lines AB and CD, Fig. 6, but such a refinement would hardly be warranted and would complicate the formula.

Using the formula given in (B) and taking $wh=p$ when $x=0$, we get $s=pa:L$.

**(E)** According to Carothers, the shearing stress produced by a uniformly loaded long strip at the rock surface is —

$$s_{xz}=\frac{p}{2}\left[sech\,\frac{\pi}{2}\,\frac{x-L}{2a}-sech\,\frac{\pi}{2}\,\frac{x+L}{2a}\right]$$

[(5), page 539.] As in this case, $n_z=n_x$, we see that $s_{xz}$ is also the principal shearing stress, $s$. The solution for our case can, therefore, be obtained by simple integration. For a triangular loading $dp=p/L\ d1$. (Fig. 6.) Differentiating $s$ with respect to $p$, and substituting for $dp$, we get —

$$ds=\frac{p}{2L}\left[sech\,\frac{\pi}{2}\,\frac{x-1}{2a}-sech\,\frac{\pi}{2}\,\frac{x+1}{2a}\right]d1$$

Hence —

$$s=\frac{p}{2L}\int_0^L\left[sech\,\frac{\pi}{2}\,\frac{x-1}{2a}-sech\,\frac{\pi}{2}\,\frac{x+1}{2a}\right]d1$$

which integrates to —

$$s=\frac{4a}{L}\,\frac{p}{\pi}\left[2\,arc\,tan\,e^{\frac{\pi}{2}\frac{x}{2a}}-arc\,tan\,e^{\frac{\pi}{2}\frac{x+L}{2a}}-arc\,tan\,e^{\frac{\pi}{2}\frac{x-L}{2a}}\right]$$

**(F)** Fig. 8B shows the theoretical effect of boundaries on the distribution of stresses caused by a uniformly loaded strip as computed by Carothers' method. The two cases considered are a rough rigid boundary surface and a surface having no shearing strength, each for depths of from $\frac{1}{4}\,b$ to $3b$ below the ground surface. The stresses have been determined at the boundary planes only, as the expressions for intermediate points are involved and lead to slowly converging series.

As is seen, the discontinuities affect both the magnitude and the distribution of stresses and cause considerably higher concentrations than occur in a homogeneous material. This is especially pronounced in the case of a frictionless layer at a moderate depth below the foundation.

Apart from considerations of the ultimate resistance against shearing failure, the effect of boundaries is of interest also when the external pressures are moderate, as in building foundations where the settlements must be small. The portion of settlement caused by distortion is very difficult to analyze quantitatively, but as it must increase with the shearing stresses it would appear that at restricted boundaries it will play a more important part than in the homogeneous case where the shearing stresses are relatively lower. As distortions are likely to cause some disturbance in the structure of the material, they might also contribute to the settlements indirectly by affecting the compressibility of the clay.

It would appear to the author that these factors might in some cases suggest the necessity of caution when it is contemplated to reduce the pressure to be imposed on an underlying soft stratum by additional excavation made to compensate in part for building loads.

To illustrate the effect of discontinuities when they occur at a greater depth, consider the $n_z$ on the vertical axis when the boundary is at such depth that $\frac{b}{h}$ is small. Then, when $x=0$ —

$$n'_z = 4\frac{p}{\pi}\left[arc\ tan\frac{b}{h} - arc\ tan\frac{b}{3h} + arc\ tan\frac{b}{5h} - \ldots\right]$$

$$= \frac{p}{\pi}\frac{4h}{b}\left[1 - \tfrac{1}{3} + \tfrac{1}{5} - \ldots\right] = \frac{h}{b}p$$

$$n_z = \frac{p}{\pi}\left[\alpha + sin\ \alpha\ cos\ 2\ \beta\right] = \frac{p}{\pi}\ 2\ arc\ tan\frac{2b}{h} = \frac{4}{\pi}\frac{b}{h}p$$

$$n''_z = -n''_x = s'' = p\ tanh\frac{\pi}{2}\frac{b}{h} = \frac{\pi}{2}\frac{b}{h}p$$

We thus get for comparison —

$$n'_z : n_z : n''_z = 1 : \frac{4}{\pi} : \frac{\pi}{2} = \frac{\pi}{4} : 1 : \frac{\pi^2}{8} = .786 : 1 : 1.23$$

Compared with the homogeneous material the rock reduces and a frictionless layer increases the stress concentration by about 25 per cent.

The stresses at the rigid boundary do not agree with values obtained by Melan in Beton *u.* Eisen, Heft 7, 1919, page 83. Melan's derivation rests on the assumption that the case is equivalent to one half of a plate with equal symmetrical loadings on both surfaces, with the rigid boundary corresponding to the central plane, where the vertical movements are zero due to symmetry. It should be noted, however, that the lateral movements on this plane are not restricted; consequently, it would appear to the author that the two cases are not actually equivalent.

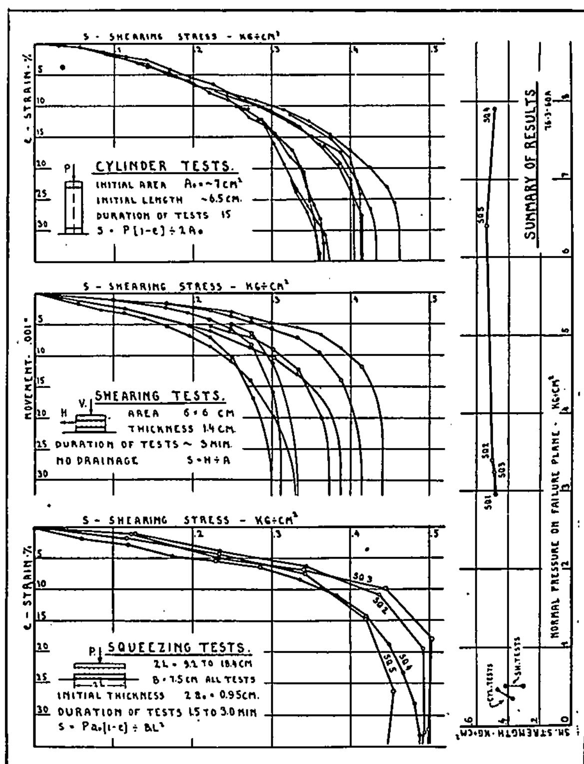


FIG. 7.— TEST RESULTS ON TYPICAL CLAY SAMPLE

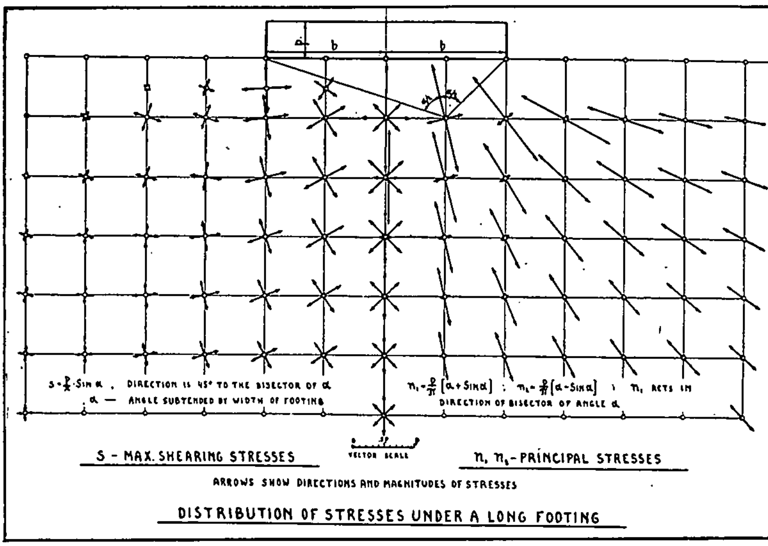

b
b
$S = \frac{p}{\pi} \cdot \sin\alpha$, DIRECTION IS 45° TO THE BISECTOR OF $\alpha$
$\alpha$ — ANGLE SUBTENDED BY WIDTH OF FOOTING
$n_1 = \frac{p}{\pi}[\alpha + \sin\alpha]$ ; $n_2 = \frac{p}{\pi}[\alpha - \sin\alpha]$ ; $n_1$ ACTS IN DIRECTION OF BISECTOR OF ANGLE $\alpha$
S - MAX. SHEARING STRESSES
VECTOR SCALE
$n_1$ $n_2$ - PRINCIPAL STRESSES
ARROWS SHOW DIRECTIONS AND MAGNITUDES OF STRESSES
DISTRIBUTION OF STRESSES UNDER A LONG FOOTING

FIG. 8

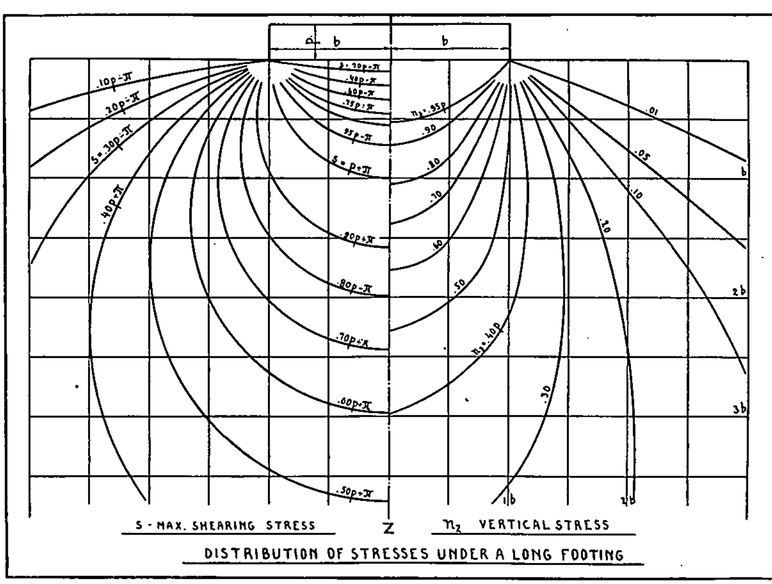
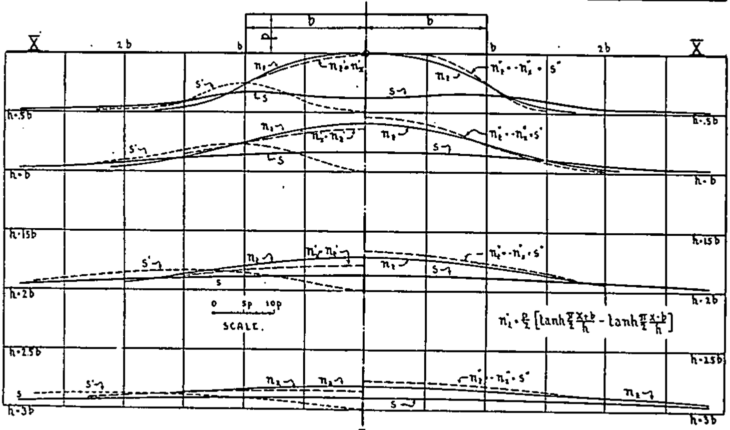

b
b
.10p÷π
.20p÷π
S=.30p÷π
.40p÷π
.95p÷π
S=p÷π
.90p÷π
.80p÷π
.70p÷π
.60p÷π
.50p÷π
$n_2$=.95p
.90
.80
.70
.60
.50
$n_2$=.40p
.30
.20
.10
.05
.01
b
2b
3b
1b
2b
S - MAX. SHEARING STRESS
Z
$n_2$ VERTICAL STRESS
DISTRIBUTION OF STRESSES UNDER A LONG FOOTING

FIG. 8A

| h | h = 0.25b | | | | | h = 0.50b | | | | | h = b | | | | | h = 2b | | | | | h = 3b | | | | | h |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| x/b | 0 | .55 | 1. | 1.5 | 2.0 | 0 | .5 | 1 | 1.5 | 2.0 | 0 | 0.5 | 1 | 1.5 | 2.0 | 0 | 0.5 | 1 | 1.5 | 2.0 | 0 | 0.5 | 1.0 | 1.5 | 2.0 | x/b |
| $n_z'$ | 1.00 | .98 | .472 | .131 | -.010 | .944 | .855 | .489 | | .011 | .739 | .661 | 446 | | .117 | .450 | | .366 | | .147 | .314 | | .313 | | .213 | $n_z'$ |
| $n_z$ | 1.00 | .98 | .500 | .018 | .003 | .959 | .903 | .497 | .089 | .019 | .818 | .735 | .480 | .249 | .078 | .551 | .511 | .409 | .288 | .185 | .395 | .380 | .340 | .280 | .210 | $n_z$ |
| $n_z''$ | 1.00 | .998 | .500 | .002 | 0 | .996 | 959 | .500 | .036 | .002 | .917 | .819 | .498 | .172 | .041 | .656 | .600 | .459 | .294 | .163 | .480 | .451 | .390 | .159 | .220 | $n_z''$ |
| $s'$ | 0 | .043 | .500 | .043 | .002 | 0 | .191 | .498 | .198 | .043 | 0 | .280 | .457 | .358 | .190 | 0 | .195 | .301 | | .283 | 0 | | .188 | | .240 | $s'$ |
| $s$ | .150 | .187 | .316 | .113 | .051 | .255 | .285 | .309 | .177 | .094 | .318 | .316 | .284 | .212 | .142 | .255 | .247 | .225 | .193 | .158 | .191 | .19 | .175 | .152 | .143 | $s$ |
| $s''$ | 1.00 | .998 | .500 | .002 | 0 | .996 | .959 | .500 | .036 | .002 | .917 | .819 | .498 | .172 | .041 | .656 | .600 | .459 | .294 | .163 | .484 | .451 | .390 | .159 | .220 | $s''$ |




$n_z'$ $s$ - STRESSES AT RIGID BOUNDARY

$n_z''$ $s''$ - STRESSES AT FRICTIONLESS BOUNDARY

SOLID LINES REFER TO HOMOGENEOUS MATERIAL · $n_z$ $s$

EFFECT OF DISCONTINUITIES

FIG. 8B

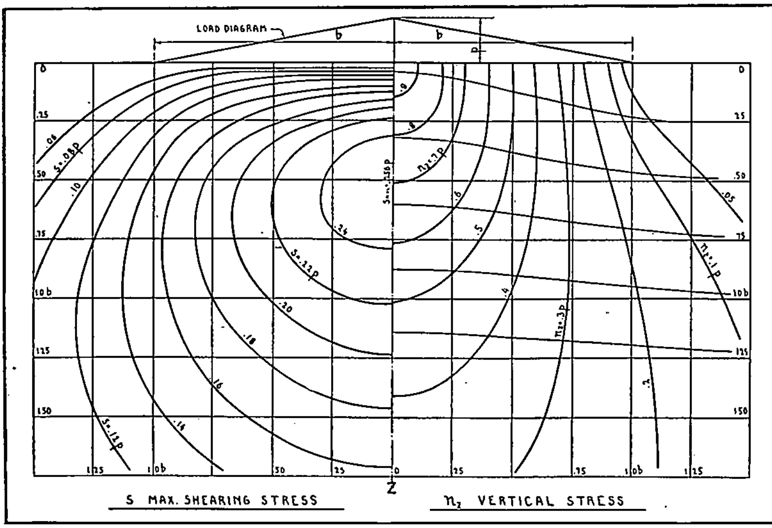

LOAD DIAGRAM
S MAX. SHEARING STRESS
$n_z$ VERTICAL STRESS

FIG. 9

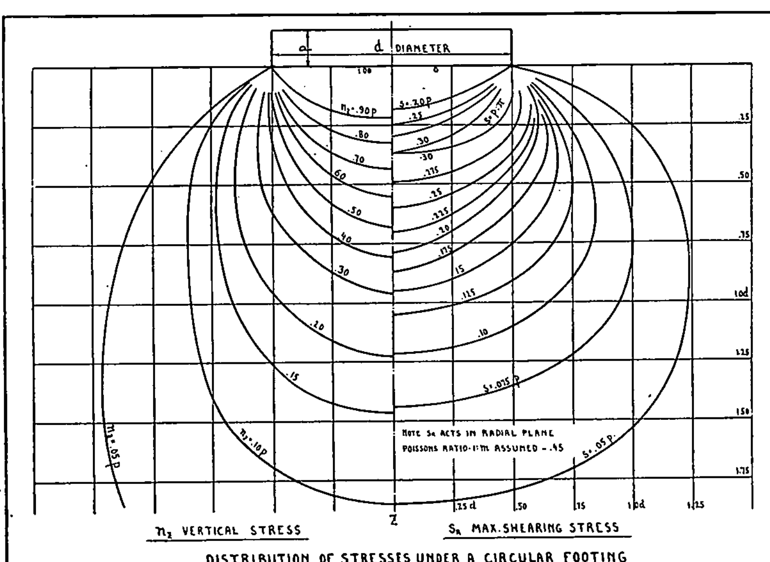

d DIAMETER
NOTE $S_R$ ACTS IN RADIAL PLANE
POISSONS RATIO-1:m ASSUMED -.45
$n_z$ VERTICAL STRESS
$S_R$ MAX. SHEARING STRESS
DISTRIBUTION OF STRESSES UNDER A CIRCULAR FOOTING

FIG. 10

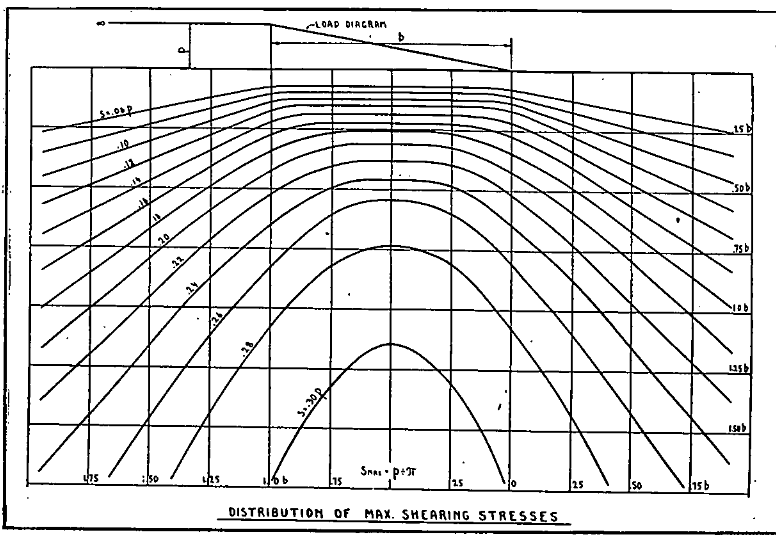



FIG. 11

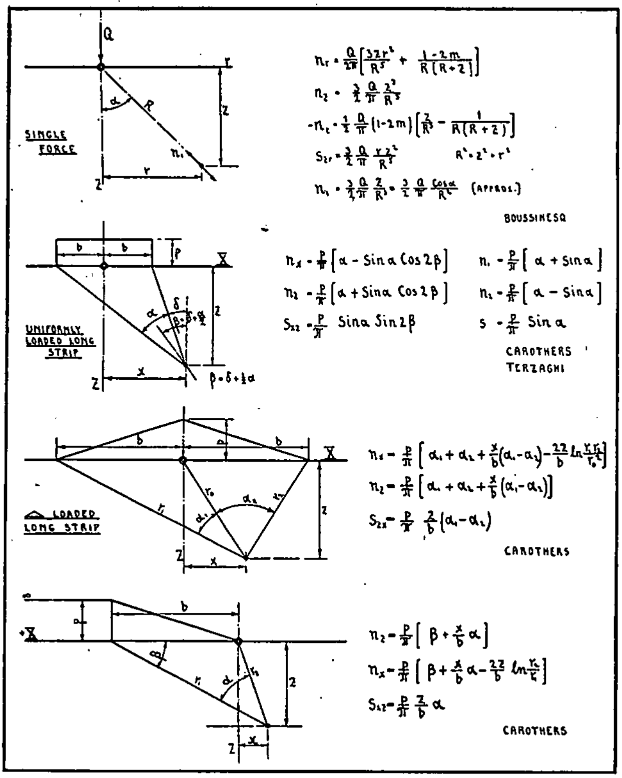

SINGLE FORCE
BOUSSINESQ
UNIFORMLY LOADED LONG STRIP
CAROTHERS
TERZAGHI
LOADED LONG STRIP
CAROTHERS
CAROTHERS

FIG. 12

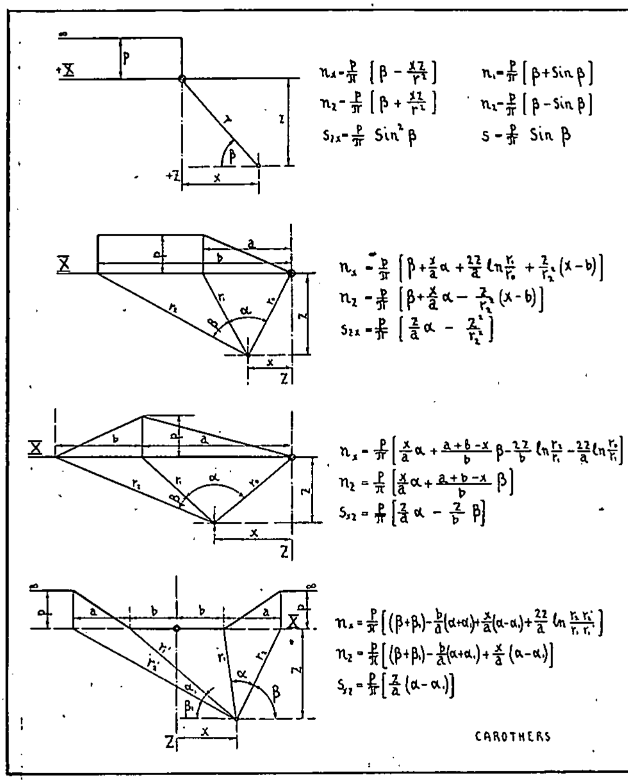



FIG. 13

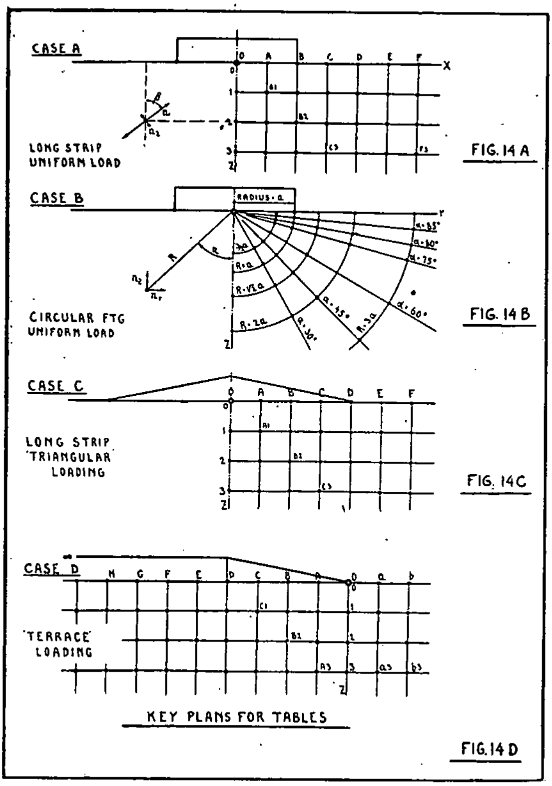

CASE A
LONG STRIP
UNIFORM LOAD
FIG. 14 A
CASE B
RADIUS = a
CIRCULAR FTG
UNIFORM LOAD
FIG. 14 B
CASE C
LONG STRIP
'TRIANGULAR'
LOADING
FIG. 14C
CASE D
'TERRACE'
LOADING
KEY PLANS FOR TABLES
FIG. 14 D

FIG. 14

## Appendix II

To illustrate the use of formulæ and tables given below it may be well to discuss briefly a typical case. Consider the case of a long footing of width $2b$, formulæ for which are given in Fig. 12. As seen, the principal stresses are functions of angle $\alpha$ only. The lines of equal intensity of all principal stresses are therefore circles passing through the edges of the loaded area, as is shown in Fig. 8A for the principal shearing stresses. Vectors in Fig. 8 show the directions and intensities of stresses at the given point. The major principal stress ($n_1$) acts in the direction of the bisector of c. The minor principal stress ($n_2$) is at 90° and the maximum shearing stresses ($s$) at 45° to the major. On the central axis, $n_1 = n_z$, $n_2 = n_x$ and the maximum shearing stresses act in 45° planes to the vertical. The value of $s$ is highest when $\alpha = 90°$, *i.e.*, on the semicircle as shown in Figs. 8 and 8A.

The intensity of $n_2$ drops rapidly with $\alpha$. Thus, if the loaded strip is relatively narrow there remains but one principal normal stress acting in the direction of the loaded area. The intensity is $n_1 = p\ (\alpha + sin\ \alpha)/\pi = (2p\ .\ 2b\ (cos^2\ \beta)/\pi\ z$ when $\alpha$ approaches zero. The lines of equal intensity are then circles tangent to the point of application.

Table B is taken from Carothers' paper before the Mathematical Congress in Toronto. The values of $s$ were computed from the data given by Carothers. Tables C and D were prepared by assistants furnished by the Boston Engineers' Emergency Planning Bureau. As used in the tables, $\beta$ is the angle between the direction of $n_1$ and the vertical.

Case A. Long Strip. — Uniform Loading

[Key plan, Fig. 14A]

| Point | $n_z$ | $n_x$ | $s_{zx}$ | $\beta$ | $s_{max}$ | $n_1$ | $n_2$ |
|---|---|---|---|---|---|---|---|
| 00 | 1.0000 | 1.0000 | 0 | 0 | 0 | 1.0000 | 1.0000 |
| 01 | .9594 | .4498 | 0 | 0 | .2548 | .9594 | .4498 |
| 02 | .8183 | .1817 | 0 | 0 | .3183 | .8183 | .1817 |
| 03 | .6678 | .0803 | 0 | 0 | .2937 | .6678 | .0803 |
| 04 | .5508 | .0026 | 0 | 0 | .2546 | .5498 | .0410 |
| 05 | .4617 | .0228 | 0 | 0 | .2195 | .4616 | .0228 |
| 06 | .3954 | .0138 | 0 | 0 | .1908 | .3954 | .0138 |
| 07 | .3457 | .0091 | 0 | 0 | .1683 | .3457 | .0091 |
| 08 | .3050 | .0061 | 0 | 0 | .1499 | .3050 | .0061 |
| A0 | 1.0000 | 1.0000 | 0 | 0 | 0 | 1.0000 | 1.0000 |
| A1 | .9028 | .3920 | .1274 | 13° 17′ | .2848 | .9323 | .3629 |
| A2 | .7352 | .1863 | .1590 | 14° 52′ | .3158 | .7763 | .1446 |
| A3 | .6073 | .0994 | .1275 | 13° 18′ | .2847 | .6370 | .0677 |
| A4 | .5107 | .0542 | .0959 | 11° 25′ | .2470 | .5298 | .0357 |
| A5 | .4372 | .0334 | .0721 | 9° 49′ | .2143 | .4693 | .0206 |
| B1 | .4969 | .3472 | .2996 | 37° 59′ | .3088 | .7308 | .1133 |
| B2 | .4797 | .2250 | .2546 | 31° 43′ | .2847 | .6371 | .0677 |
| B3 | .4480 | .1424 | .2037 | 26° 34′ | .2546 | .5498 | .0406 |
| B4 | .4091 | .0908 | .1592 | 22° 30′ | .2251 | .4751 | .0249 |
| B5 | .3701 | .0595 | .1243 | 19° 20′ | .1989 | .4137 | .0159 |
| C1 | .0892 | .2850 | .1466 | 61° 50′ | .1765 | .3636 | .0106 |
| C2 | .2488 | .2137 | .2101 | 47° 23′ | .2115 | .4428 | .0198 |
| C3 | .2704 | .1807 | .2022 | 38° 44′ | .2071 | .4327 | .0184 |
| C4 | .2876 | .1268 | .1754 | 32° 41′ | .1929 | .4007 | .0143 |
| C5 | .2851 | .0892 | .1469 | 28° 09′ | .1765 | .3637 | .0106 |
| D1 | .0194 | .1714 | .0552 | 71° 59′ | .0940 | .1893 | .0014 |
| D2 | .0776 | .2021 | .1305 | 58° 17′ | .1424 | .2834 | .0052 |
| D3 | .1458 | .1847 | .1568 | 48° 32′ | .1578 | .3232 | .0074 |
| D4 | .1847 | .1456 | .1567 | 41° 27′ | .1579 | .3232 | .0073 |
| D5 | .2045 | .1256 | .1442 | 36° 02′ | .1515 | .3094 | .0064 |
| E1 | .0060 | .1104 | .0254 | 76° 43′ | .0569 | .1141 | .0003 |
| E2 | .0357 | .1615 | .0739 | 65° 12′ | .0970 | .1957 | .0016 |
| E3 | .0771 | .1645 | .1096 | 55° 52′ | .1180 | .2388 | .0031 |
| E4 | .1135 | .1447 | .1258 | 48° 32′ | .1265 | .2556 | .0026 |
| E5 | .1404 | .1205 | .1266 | 42° 45′ | .1269 | .2575 | .0036 |

CASE A. LONG STRIP. — UNIFORM LOADING — *Concluded*

| Point | $n_z$ | $n_x$ | $s_{zx}$ | $\beta$ | $s_{max}$ | $n_1$ | $n_2$ |
|---|---|---|---|---|---|---|---|
| F1 | .0026 | .0741 | .0137 | 79° 25′ | .0379 | .0758 | .0001 |
| F2 | .0171 | .1221 | .0449 | 69° 42′ | .0690 | .1384 | .0005 |
| F3 | .0427 | .1388 | .0757 | 61° 15′ | .0895 | .1803 | .0012 |
| F4 | .0705 | .1341 | .0954 | 54° 12′ | .1006 | .2029 | .0018 |
| F5 | .0952 | .1196 | .1036 | 48° 20′ | .1054 | .2128 | .0020 |
| F6 | .1139 | .1019 | .1057 | 43° 22′ | .1058 | .2137 | .0020 |
| A½ | .9787 | .6214 | .0552 | 8° 35′ | .1871 | .9871 | .6129 |
| B½ | .4996 | .4208 | .3134 | 41° 25′ | .3158 | .7760 | .1444 |
| C½ | .0177 | .2079 | .0606 | 73° 47′ | .1128 | .2281 | .0025 |
| D½ | .0027 | .0987 | .0164 | 80° 35′ | .0507 | .1014 | .0002 |

CASE B. CIRCULAR AREA. — UNIFORM LOAD

[Key plan, Fig. 14B]

| $\alpha$ | $n_r$ | $n_\theta$ | $n_z$ | $s_{rz}$ | $s$ |
|---|---|---|---|---|---|
| R = 0 | −.9500 | −.9500 | −1.0000 | .0000 | |
| $R = 2a/3$ | | | | | |
| 0° | .2310 | .2310 | .7904 | .0000 | .280 |
| 30° | .2548 | .2625 | .8376 | .0637 | .298 |
| 45° | .3344 | .3243 | .8585 | .0529 | .267 |
| 60° | .4129 | .4252 | .9062 | .0500 | .252 |
| 80° | .6910 | .7213 | .9910 | .0270 | .152 |
| 85° | .8134 | .8324 | .9991 | .0071 | .093 |
| 90° | .9500 | .9500 | 1.0000 | .0000 | .025 |
| $R = a$ | | | | | |
| 0° | .1016 | .1016 | .6466 | .0000 | .273 |
| 30° | .1406 | .1121 | .6283 | .1364 | .280 |
| 45° | .1903 | .1409 | .6064 | .1980 | .287 |
| 60° | .2607 | .1879 | .5769 | .2533 | .299 |
| 75° | .3516 | .2676 | .5406 | .2996 | .314 |
| 80° | .3852 | .3201 | .5272 | .3075 | .316 |
| 85° | .4194 | .3935 | .5127 | .3136 | .317 |
| 90° | .4500 | .5000 | .5000 | .3162 | .317 |

CASE B. CIRCULAR AREA. — UNIFORM LOAD — *Concluded*

| $\alpha$ | $n_r$ | $n_\theta$ | $n_z$ | $s_{rz}$ | $s$ |
|---|---|---|---|---|---|
| $R=\sqrt{2}a$ | | | | | |
| 0° | .0265 | .0265 | .4810 | .0000 | .227 |
| 30° | .0944 | .0430 | .3979 | .1306 | .200 |
| 45° | .1337 | .0470 | .3442 | .1599 | .192 |
| 60° | .2144 | .0533 | .2117 | .1906 | .191 |
| 75° | .2138 | .0498 | .0694 | .1112 | .133 |
| 80° | .1564 | .0435 | .0384 | .0579 | .083 |
| 85° | .0732 | .0348 | .0164 | .0160 | .033 |
| 90° | −.0250 | .0250 | .0000 | .0000 | .013 |
| $R=2a$ | | | | | |
| 0° | .0119 | .0113 | .2825 | .0000 | .135 |
| 30° | .0538 | .0118 | .2234 | .1015 | .132 |
| 45° | .0904 | .0128 | .1583 | .1200 | .125 |
| 75° | .0932 | .0133 | .0200 | .0373 | .052 |
| 80° | .0660 | .0129 | .0037 | .0181 | .035 |
| 85° | .0257 | .0126 | .0013 | .0047 | .012 |
| 90° | −.0125 | .0125 | .0000 | .0000 | .006 |
| $R=3a$ | | | | | |
| 0° | .0013 | .0013 | .1463 | .0000 | .073 |
| 30° | .0273 | .0013 | .1052 | .0534 | .066 |
| 45° | .0482 | .0024 | .0643 | .0560 | .057 |
| 60° | .0575 | .0031 | .0256 | .0388 | .043 |
| 75° | .0361 | .0044 | .0033 | .0128 | .022 |
| 80° | .0241 | .0046 | .0018 | .0065 | .013 |
| 85° | .0090 | .0051 | .0000 | .0015 | .005 |
| 90° | −.0055 | .0055 | .0000 | .0000 | .003 |
| $R=4a$ | | | | | |
| 0° | .0008 | .0000 | .0863 | .0000 | .043 |
| 30° | .0160 | .0002 | .0603 | .0325 | .040 |
| 45° | .0317 | .0005 | .0313 | .0324 | .032 |
| 60° | .0327 | .0011 | .0192 | .0202 | .021 |
| 75° | .0203 | .0020 | .0017 | .0067 | .011 |
| 80° | .0136 | .0023 | .0002 | .0031 | .007 |
| 85° | .0052 | .0027 | .0001 | .0008 | .003 |
| 90° | −.0031 | .0031 | .0000 | .0000 | .002 |

CASE C. LONG STRIP. — TRIANGULAR LOADING

[Key plan, Fig. 14C]

| Point | $n_z$ | $n_x$ | $s_{zx}$ | $\beta$ | $s_{max}$ | $n_1$ | $n_2$ |
|---|---|---|---|---|---|---|---|
| 00 | 1.0000 | 1.0000 | 0 | 0 | 0 | 1.0000 | 1.0000 |
| 01 | 0.8440 | 0.3931 | 0 | 0 | 0.2255 | 0.8440 | 0.3931 |
| 02 | 0.7048 | 0.1925 | 0 | 0 | 0.2562 | 0.7048 | 0.1925 |
| 03 | 0.5904 | 0.1025 | 0 | 0 | 0.2440 | 0.5904 | 0.1025 |
| 04 | 0.5000 | 0.0588 | 0 | 0 | 0.2206 | 0.5000 | 0.0588 |
| 05 | 0.4296 | 0.0359 | 0 | 0 | 0.1969 | 0.4296 | 0.0359 |
| 06 | 0.3744 | 0.0234 | 0 | 0 | 0.1755 | 0.3744 | 0.0234 |
| 07 | 0.3305 | 0.0158 | 0 | 0 | 0.1574 | 0.3305 | 0.0158 |
| 08 | 0.2952 | 0.0111 | 0 | 0 | 0.1421 | 0.2952 | 0.0111 |
| 010 | 0.2422 | 0.0062 | 0 | 0 | 0 | 0.1242 | 0.1242 |
| A0 | 0.7500 | 0.7500 | 0 | – | 0 | 0.7500 | 0.7500 |
| A1 | 0.7196 | 0.3874 | 0.1151 | 17° 22′ | 0.2021 | 0.7556 | 0.3514 |
| A2 | 0.6344 | 0.2026 | 0.1146 | 13° 59′ | 0.2444 | 0.6629 | 0.1741 |
| A3 | 0.5462 | 0.1138 | 0.0951 | 11° 53′ | 0.2361 | 0.5661 | 0.0939 |
| A4 | 0.4711 | 0.0681 | 0.0756 | 10° 17′ | 0.2152 | 0.4848 | 0.0544 |
| A5 | 0.4101 | 0.0432 | 0.0597 | 9° 01′ | 0.1930 | 0.4197 | 0.0337 |
| B0 | 0.5000 | 0.5000 | 0 | – | 0 | 0.5000 | 0.5000 |
| B1 | 0.4949 | 0.3357 | 0.1525 | 31° 13′ | 0.1720 | 0.5873 | 0.2433 |
| B2 | 0.4714 | 0.2152 | 0.1762 | 27° 00′ | 0.2178 | 0.5611 | 0.1255 |
| B3 | 0.4350 | 0.1385 | 0.1570 | 23° 19′ | 0.2160 | 0.5028 | 0.0708 |
| B4 | 0.3955 | 0.0913 | 0.1299 | 20° 15′ | 0.2000 | 0.4434 | 0.0434 |
| B5 | 0.3577 | 0.0617 | 0.1055 | 17° 45′ | 0.1817 | 0.3914 | 0.0280 |
| B6 | 0.3238 | 0.0433 | 0.0858 | 15° 44′ | 0.1644 | 0.3480 | 0.0192 |
| B8 | 0.2682 | 0.0229 | 0.0582 | 12° 42′ | 0.1358 | 0.2814 | 0.0098 |
| B10 | 0.2266 | 0.0130 | 0.0415 | 10° 37′ | 0.1146 | 0.2344 | 0.0052 |
| C0 | 0.2500 | 0.2500 | 0 | – | 0 | 0.2500 | 0.2500 |
| C1 | 0.2620 | 0.2620 | 0.1476 | 45° 00′ | 0.1476 | 0.4096 | 0.1144 |
| C2 | 0.2875 | 0.2162 | 0.1810 | 39° 26′ | 0.1845 | 0.4364 | 0.0674 |
| C3 | 0.3000 | 0.1611 | 0.1735 | 34° 05′ | 0.1869 | 0.4175 | 0.0437 |
| C4 | 0.2980 | 0.1167 | 0.1528 | 29° 39′ | 0.1777 | 0.3851 | 0.0297 |
| C5 | 0.2869 | 0.0847 | 0.1309 | 26° 10′ | 0.1654 | 0.3512 | 0.0204 |
| D0 | 0 | 0 | 0 | – | 0 | 0 | 0 |
| D1 | 0.0766 | 0.1956 | 0.0959 | 60° 54′ | 0.1129 | 0.0232 | 0.2490 |
| D2 | 0.1393 | 0.2005 | 0.1414 | 51° 06′ | 0.1447 | 0.0252 | 0.3146 |
| D3 | 0.1813 | 0.1693 | 0.1534 | 43° 53′ | 0.1535 | 0.0218 | 0.3288 |

CASE C. LONG STRIP. — TRIANGULAR LOADING — *Concluded*

| Point | $n_z$ | $n_x$ | $s_{zx}$ | $\beta$ | $s_{max}$ | $n_1$ | $n_2$ |
|---|---|---|---|---|---|---|---|
| D4 | 0.2048 | 0.1338 | 0.1476 | 38° 14′ | 0.1518 | 0.0175 | 0.3211 |
| D5 | 0.2148 | 0.1033 | 0.1343 | 33° 43′ | 0.1454 | 0.0137 | 0.3045 |
| D6 | 0.2159 | 0.0794 | 0.1189 | 30° 04′ | 0.1371 | 0.0106 | 0.2848 |
| D7 | 0.2048 | 0.0471 | 0.0903 | 29° 25′ | 0.1199 | 0.0061 | 0.2459 |
| D10 | 0.1874 | 0.0298 | 0.0685 | 20° 30′ | 0.1044 | 0.0042 | 0.2130 |
| E1 | 0.0155 | 0.1278 | 0.0399 | 72° 18′ | 0.0689 | 0.0028 | 0.1406 |
| E2 | 0.0580 | 0.1668 | 0.0899 | 60° 35′ | 0.1051 | 0.0073 | 0.2175 |
| E3 | 0.1002 | 0.1599 | 0.1169 | 52° 10′ | 0.1207 | 0.0094 | 0.2508 |
| E4 | 0.1319 | 0.1379 | 0.1256 | 45° 41′ | 0.1256 | 0.0093 | 0.2605 |
| E5 | 0.1526 | 0.1137 | 0.1232 | 40° 30′ | 0.1247 | 0.0085 | 0.2579 |
| F1 | 0.0046 | 0.0840 | 0.0186 | 77° 27′ | 0.0439 | 0.0004 | 0.0882 |
| F2 | 0.0250 | 0.1294 | 0.0540 | 67° 01′ | 0.0751 | 0.0021 | 0.1523 |
| F3 | 0.0545 | 0.1396 | 0.0828 | 58° 37′ | 0.0934 | 0.0037 | 0.1905 |
| F4 | 0.0824 | 0.1315 | 0.0992 | 51° 58′ | 0.1022 | 0.0048 | 0.2092 |
| F5 | 0.1049 | 0.1156 | 0.1053 | 46° 28′ | 0.1054 | 0.0049 | 0.2157 |
| F6 | 0.1211 | 0.0981 | 0.1044 | 41° 51′ | 0.1050 | 0.0046 | 0.2146 |
| F8 | 0.1375 | 0.0681 | 0.0929 | 34° 45′ | 0.0992 | 0.0036 | 0.2020 |
| F10 | 0.1403 | 0.0470 | 0.0781 | 29° 33′ | 0.0910 | 0.0027 | 0.1847 |
| H2 | 0.0064 | 0.7773 | 0.0222 | 74° 02′ | 0.0420 | 0.0841 | 0.0000 |
| H4 | 0.0332 | 0.1041 | 0.0572 | 60° 54′ | 0.0673 | 0.0014 | 0.1360 |
| H6 | 0.0636 | 0.0955 | 0.0763 | 50° 55′ | 0.0780 | 0.0016 | 0.1576 |
| H8 | 0.0862 | 0.0761 | 0.0791 | 43° 10′ | 0.0793 | 0.1605 | 0.0019 |
| H10 | 0.0985 | 0.0581 | 0.0741 | 37° 23′ | 0.0768 | 0.1551 | 0.0015 |

## CASE D. "TERRACE" LOADING

[Key plan, Fig. 14D]

| Point | $n_z$ | $n_x$ | $s_{zx}$ | $90°-\beta$ | $s_{max}$ | $n_1$ | $n_2$ |
|---|---|---|---|---|---|---|---|
| 00 | 0 | 0 | 0 | – | 0 | 0 | 0 |
| 01 | 0.0780 | 0.3034 | 0.1055 | 21° 33′ | 0.1544 | 0.3451 | 0.0363 |
| 02 | 0.1476 | 0.4038 | 0.1762 | 27° 00′ | 0.2178 | 0.4935 | 0.0579 |
| 03 | 0.2048 | 0.4487 | 0.2214 | 30° 35′ | 0.2528 | 0.5796 | 0.0740 |
| 04 | 0.2500 | 0.4706 | 0.2500 | 33° 06′ | 0.2732 | 0.6335 | 0.0871 |
| 05 | 0.2852 | 0.4821 | 0.2685 | 34° 56′ | 0.2860 | 0.6697 | 0.0977 |
| 06 | 0.3129 | 0.4883 | 0.2807 | 36° 20′ | 0.2941 | 0.6947 | 0.1065 |
| 08 | 0.3524 | 0.4946 | 0.2952 | 38° 14′ | 0.3036 | 0.7271 | 0.1199 |
| 010 | 0.3789 | 0.4968 | 0.3028 | 39° 30′ | 0.3085 | 0.7464 | 0.1294 |
| A0 | 0.2500 | 0.2500 | 0 | – | 0 | 0.2500 | 0.2500 |
| A1 | 0.2643 | 0.3924 | 0.1619 | 34° 13′ | 0.1741 | 0.5025 | 0.1543 |
| A2 | 0.3023 | 0.4544 | 0.2302 | 35° 52′ | 0.2424 | 0.6208 | 0.1360 |
| A3 | 0.3381 | 0.4784 | 0.2643 | 37° 34′ | 0.2734 | 0.6817 | 0.1349 |
| A4 | 0.3659 | 0.4885 | 0.2828 | 38° 53′ | 0.2894 | 0.7166 | 0.1378 |
| A5 | 0.3867 | 0.4935 | 0.2936 | 39° 51′ | 0.2984 | 0.7385 | 0.1417 |
| a0 | 0 | 0 | 0 | – | 0 | 0 | 0 |
| a1 | 0.0161 | 0.2201 | 0.0468 | 12° 19′ | 0.1123 | 0.2304 | 0.0058 |
| a2 | 0.0633 | 0.3430 | 0.1157 | 19° 48′ | 0.1816 | 0.3848 | 0.0216 |
| a3 | 0.1156 | 0.4077 | 0.1692 | 24° 36′ | 0.2235 | 0.4852 | 0.0382 |
| a4 | 0.1630 | 0.4431 | 0.2072 | 27° 59′ | 0.2501 | 0.5532 | 0.0530 |
| a5 | 0.2032 | 0.4634 | 0.2339 | 30° 28′ | 0.2677 | 0.6010 | 0.0656 |
| B0 | 0.5000 | 0.5000 | 0 | – | 0 | 0.5000 | 0.5000 |
| B1 | 0.5000 | 0.5000 | 0.1762 | 45° 00′ | 0.1762 | 0.6762 | 0.3238 |
| B2 | 0.5000 | 0.5000 | 0.2500 | 45° 00′ | 0.2500 | 0.7500 | 0.2500 |
| B3 | 0.5000 | 0.5000 | 0.2807 | 45° 00′ | 0.2807 | 0.7807 | 0.2193 |
| B4 | 0.5000 | 0.5000 | 0.2952 | 45° 00′ | 0.2952 | 0.7952 | 0.2048 |
| B5 | 0.5000 | 0.5000 | 0.3028 | 45° 00′ | 0.3028 | 0.8028 | 0.1972 |
| B6 | 0.5000 | 0.5000 | 0.3072 | 45° 00′ | 0.3072 | 0.8072 | 0.1928 |
| B8 | 0.5000 | 0.5000 | 0.3119 | 45° 00′ | 0.3119 | 0.8119 | 0.1881 |
| B10 | 0.5000 | 0.5000 | 0.3144 | 45° 00′ | 0.3144 | 0.8144 | 0.1856 |

CASE D. "TERRACE" LOADING — *Continued*

| Point | $n_z$ | $n_x$ | $s_{zx}$ | $90°-\beta$ | $s_{max}$ | $n_1$ | $n_2$ |
|---|---|---|---|---|---|---|---|
| b0 | 0 | 0 | 0 | – | 0 | 0 | 0 |
| b1 | 0.0051 | 0.1642 | 0.0237 | 8° 19′ | 0.0829 | 0.1676 | 0.0018 |
| b2 | 0.0286 | 0.2847 | 0.0738 | 14° 59′ | 0.1478 | 0.3045 | 0.0089 |
| b3 | 0.0650 | 0.3614 | 0.1239 | 19° 57′ | 0.1932 | 0.4064 | 0.0200 |
| b4 | 0.1045 | 0.4086 | 0.1653 | 23° 42′ | 0.2246 | 0.4812 | 0.0320 |
| b5 | 0.1422 | 0.4381 | 0.1972 | 26° 41′ | 0.2465 | 0.5367 | 0.0437 |
| b6 | 0.1762 | 0.4566 | 0.2214 | 28° 50′ | 0.2621 | 0.5785 | 0.0543 |
| b8 | 0.2317 | 0.4771 | 0.2537 | 32° 06′ | 0.2818 | 0.6362 | 0.0726 |
| b10 | 0.2734 | 0.4869 | 0.2729 | 34° 19′ | 0.2930 | 0.6732 | 0.0872 |
| C0 | 0.7500 | 0.7500 | 0 | – | 0 | 0.7500 | 0.7500 |
| C1 | 0.7357 | 0.6076 | 0.1619 | 55° 47′ | 0.1741 | 0.8458 | 0.4976 |
| C2 | 0.6978 | 0.5457 | 0.2302 | 54° 08′ | 0.2424 | 0.8642 | 0.3794 |
| C3 | 0.6619 | 0.5217 | 0.2643 | 52° 26′ | 0.2734 | 0.8652 | 0.3184 |
| C4 | 0.6341 | 0.5114 | 0.2828 | 51° 07′ | 0.2894 | 0.8622 | 0.2834 |
| C5 | 0.6133 | 0.5065 | 0.2936 | 50° 09′ | 0.2984 | 0.8583 | 0.2615 |
| c0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| c1 | 0.0023 | 0.1302 | 0.0143 | 6° 18′ | 0.0656 | 0.1319 | 0.0007 |
| c2 | 0.0147 | 0.2382 | 0.0493 | 11° 54′ | 0.1222 | 0.2487 | 0.0043 |
| D0 | 1.0000 | 1.0000 | 0 | 90° 00′ | 0 | 1.0000 | 1.0000 |
| D1 | 0.9220 | 0.6973 | 0.1055 | 68° 24′ | 0.1541 | 0.9638 | 0.6556 |
| D2 | 0.8524 | 0.5962 | 0.1762 | 63° 01′ | 0.2178 | 0.9421 | 0.5065 |
| D3 | 0.7951 | 0.5512 | 0.2214 | 59° 26′ | 0.2528 | 0.9260 | 0.4204 |
| D4 | 0.7500 | 0.5294 | 0.2500 | 56° 54′ | 0.2732 | 0.9129 | 0.3665 |
| D5 | 0.7148 | 0.5178 | 0.2685 | 55° 05′ | 0.2860 | 0.9023 | 0.3303 |
| D6 | 0.6871 | 0.5117 | 0.2807 | 53° 41′ | 0.2941 | 0.8935 | 0.3053 |
| D8 | 0.6476 | 0.5054 | 0.2952 | 51° 47′ | 0.3036 | 0.8801 | 0.2729 |
| D10 | 0.6211 | 0.5032 | 0.3028 | 50° 31′ | 0.3085 | 0.8707 | 0.2537 |
| d0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| d2 | 0.0083 | 0.2031 | 0.0348 | 9° 50′ | 0.1034 | 0.2091 | 0.0023 |
| d4 | 0.0452 | 0.3369 | 0.1024 | 17° 33′ | 0.1782 | 0.3693 | 0.0129 |
| d6 | 0.0967 | 0.4092 | 0.1621 | 23° 02′ | 0.2252 | 0.4782 | 0.0278 |
| d8 | 0.1475 | 0.4465 | 0.2049 | 26° 57′ | 0.2536 | 0.5506 | 0.0434 |
| d10 | 0.1915 | 0.4674 | 0.2343 | 29° 45′ | 0.2719 | 0.6014 | 0.0576 |

CASE D. "TERRACE" LOADING — *Concluded*

| Point | $n_z$ | $n_x$ | $s_{zx}$ | $90°-\beta$ | $s_{max}$ | $n_1$ | $n_2$ |
|---|---|---|---|---|---|---|---|
| E0 | 1.0000 | 1.0000 | 0 | 90° 00′ | 0 | 1.0000 | 1.0000 |
| E1 | 0.9803 | 0.7763 | 0.0461 | 77° 51′ | 0.1120 | 0.9903 | 0.7663 |
| E2 | 0.9367 | 0.6570 | 0.1157 | 70° 13′ | 0.1816 | 0 9785 | 0.6153 |
| E3 | 0.8844 | 0.5923 | 0.1692 | 65° 24′ | 0.2235 | 0.9619 | 0.5149 |
| E4 | 0.8370 | 0.5569 | 0.2072 | 62° 02′ | 0.2501 | 0.9471 | 0.4469 |
| E6 | 0.7632 | 0.5247 | 0.2528 | 57° 38′ | 0.2795 | 0.9235 | 0.3645 |
| E8 | 0.7123 | 0.5124 | 0.2763 | 54° 57′ | 0.2938 | 0.9062 | 0.3186 |
| E10 | 0.6765 | 0.5069 | 0.2896 | 53° 10′ | 0.3018 | 0.8935 | 0.2899 |
| F0 | 1.0000 | 1.0000 | 0 | 90° 00′ | 0 | 1.0000 | 1.0000 |
| F1 | 0.9949 | 0.8357 | 0.0237 | 81° 42′ | 0.0829 | 0.9982 | 0.8234 |
| F2 | 0.9714 | 0.7152 | 0.0738 | 75° 02′ | 0.1478 | 0.9911 | 0.6955 |
| F3 | 0.9350 | 0.6385 | 0.1239 | 70° 03′ | 0.1932 | 0.9800 | 0.5936 |
| F4 | 0.8955 | 0.5913 | 0.1653 | 66° 19′ | 0.2246 | 0.9680 | 0.5188 |
| F6 | 0.8238 | 0.5433 | 0.2214 | 61° 11′ | 0.2621 | 0.9457 | 0.4215 |
| F8 | 0.7682 | 0.5229 | 0.2537 | 57° 54′ | 0.2818 | 0.9274 | 0.3638 |
| F10 | 0.7266 | 0.5130 | 0.2729 | 55° 41′ | 0.2930 | 0.9128 | 0.3268 |
| H0 | 1.0000 | 1.0000 | 0 | 90° 00′ | 0 | 1.0000 | 1.0000 |
| H1 | 0.9989 | 0.8919 | 0.0096 | 84° 55′ | 0.0543 | 0.9997 | 0.8911 |
| H2 | 0.9916 | 0.7968 | 0.0348 | 80° 10′ | 0.1034 | 0.9976 | 0.7908 |
| H4 | 0.9548 | 0.6631 | 0.1024 | 72° 28′ | 0.1781 | 0.9871 | 0.6309 |
| H6 | 0.9032 | 0.5908 | 0.1621 | 66° 59′ | 0.2252 | 0.9722 | 0.5218 |
| H8 | 0.8524 | 0.5534 | 0.2049 | 63° 04′ | 0.2537 | 0.9566 | 0.4492 |
| H10 | 0.8085 | 0.5325 | 0.2343 | 60° 15′ | 0.2719 | 0.9424 | 0.3986 |

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# THE SHEARING RESISTANCE OF SOILS

By Leo Jürgenson, Sc.D.*

This report presents the results, discussion of results, and deductions from research performed on natural and remoulded soil samples, using three types of shearing tests, one of which is believed to be novel. The notation and summary of main conclusions are given below.

*Notation*

The term "consolidation" is used in the sense in which it has been adopted in Soil Mechanics, — to denote the complete internal readjustment of the material to a given pressure.

The notation "$p_c$" is used to distinguish the pressure which has been maintained long enough to allow for complete consolidation.

As in Rheology, the term "deviator" is used to denote the normal stress in excess of the hydrostatic.

$p$ = external pressure.
$p_o$ = hydrostatic pressure.
$p_c$ = consolidation pressure.
$n_1$ = principal normal deviator stress.
$s$ = principal shearing stress.
$c$ = shearing strength.
$\phi$ = angle of internal friction.
$W_c$ = weight of water in per cent of weight of solid matter.
$L_o$ = initial free length of cylindrical sample.
$A_o$ = initial area of sample.
$T$ = duration of test.
$T_c$ = time allowed for consolidation.

*Summary of Results and Conclusions.*

To determine the shearing resistance of clay corresponding to its state before testing, no further consolidation should be allowed during the test. To avoid consolidation during shearing tests, drainage should be prevented, the tests should be performed rapidly and under light normal stresses.

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The simplest reliable test for determination of shearing strength is axial compression of a cylindrical specimen. This test requires no complicated apparatus, can be performed under light normal stresses, and offers an easy means for avoiding stress concentrations near the failure planes.

The higher resistances which are obtained in slow tests are caused by a partial consolidation of the material under the forces imposed in testing.

The shearing resistance depends not only upon the consolidation pressure, but also upon the internal grain structure acquired prior to the test. Depending on the structure, the shearing resistance of a typical remoulded clay (Boston Blue Clay) varies from about 0.3 to 0.6 times the consolidation pressure.

Only the lower values of this range can be relied upon for most practical applications. In Boston Blue Clay the shearing resistance in nature is generally less than 0.3 times the pressure to which it is consolidated.

The grain structure of most clays in nature is relatively very loose, and the properties are therefore very much affected by disturbances caused in sampling.

A disturbance increases the initial compressibility, decreases the shearing resistance if tested before reconsolidation, and increases the shearing strength in the reconsolidated state.

The shearing resistance of natural clays is independent of normal pressure, provided no consolidation is allowed.

There is no theoretical or experimental justification for determining the angle of internal friction of clay from orientations of failure planes on compressed cylinders.

The consolidation pressure of certain clays can be approximately determined from shearing tests on remoulded samples.

In clays possessing the property of hardening under no pressure, the interpretation of results of shearing test involves the laws of the Physics of Colloids, and thus cannot be based on laws of ordinary Mechanics only.

The shearing resistance of clays can be determined from a novel test based on Rheology and consisting of compression between rigid surfaces.

The shearing strength of cemented sands increases with the pressure at a rate which is considerably lower than the rate for uncemented sand.

## Research on Shearing Resistance of Clay.

In an attempt to clarify some of the main points of controversy on the question of shearing resistance of clays, the first series of investigations covered by this report was performed on a very uniform remoulded clay. The material used was a typical clay, the characteristics of which are given in Fig. 2. In order to have the same material for all tests, a sufficient quantity was carefully mixed at a water content slightly below the liquid limit, and stored. Samples were formed at this consistency, as it would have been difficult to handle them at the liquid limit.

Descriptions and methods of procedure for the three types of tests, which are referred to as cylinder tests, direct shearing tests and squeezing tests, will be discussed separately.

### *Cylinder Tests*

Cylindrical specimens were formed in a brass tube 31.7 mm. in diameter and 120 mm. long, taking care to avoid enclosing any air bubbles. The cylinders were removed from the tube by means of a closely fitting rubber plunger; they were wrapped in a layer of filter paper and set on a base, as shown in Fig. 1D. The sample was then covered with a thin skin of high grade rubber which was carefully sealed to the base. Consolidation to any desired pressure was accomplished by subjecting the outside of the rubber covering to a hydrostatic pressure which was maintained by a standpipe with a small water reservoir. For high pressures the standpipe was connected to an air reservoir where the pressure was kept constant by an automatic air pump. The excess water in the clay was free to escape through the filter paper, the porous disc and the drain valve in the base. The samples were kept under pressure for a period of at least two weeks, as shorter periods gave inconsistent results, and for longer periods no further effect was observed. The filter paper was removed immediately upon the release of pressure in order to prevent reabsorption of water; and although samples tested a few hours later gave identical results, most of the specimens were tested within an hour after being removed from the pressure chamber.

Prior to testing, the samples were cut square at both ends with a fine wire saw, the rubber covering was replaced, and the cylinders inserted between two brass end-pieces, as shown in Fig. 1A. All handling was done in a room where a relative humidity of 98 per cent was maintained. The ends of the specimens were reinforced by a strip of soft

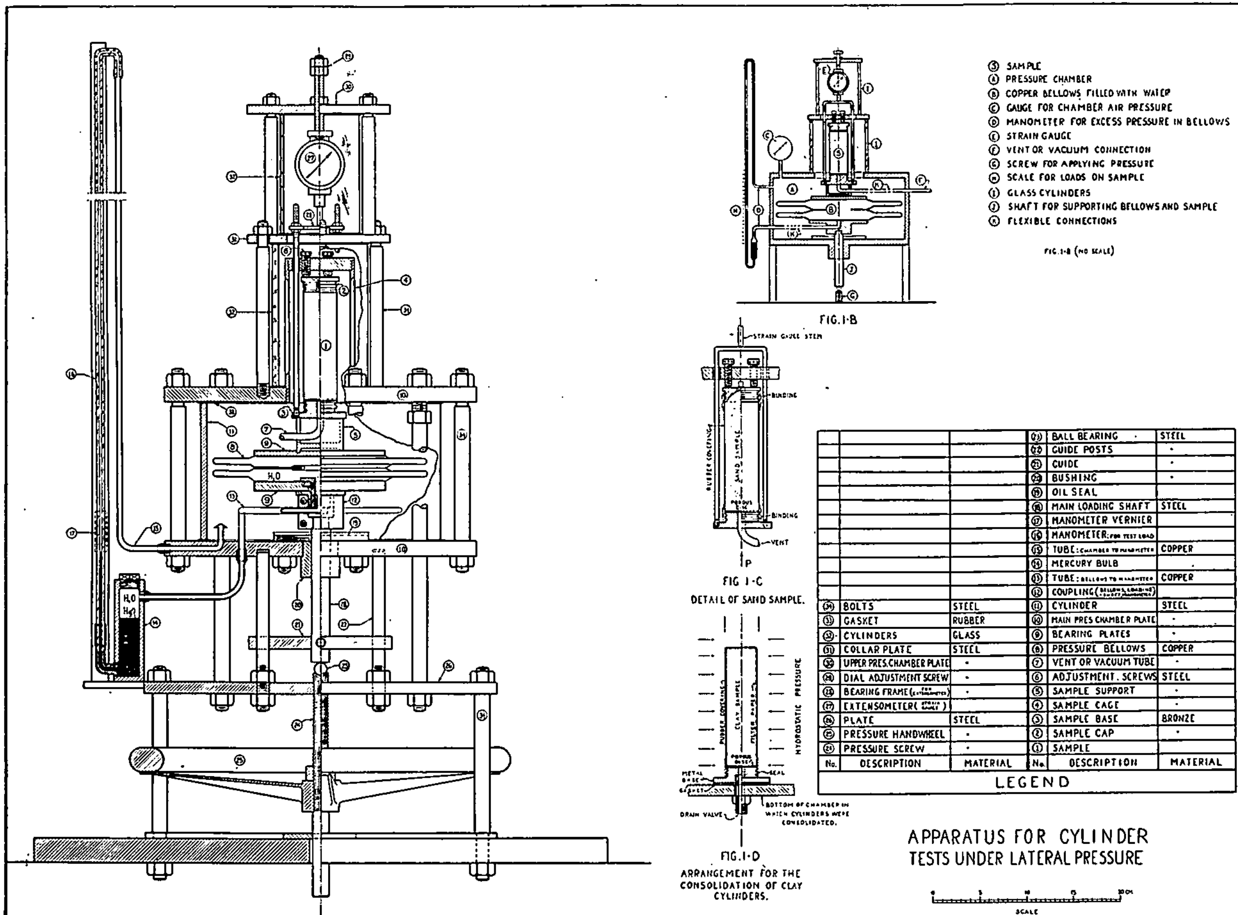

H₂O
Hg
FIG. 1-B
(S) SAMPLE
(A) PRESSURE CHAMBER
(B) COPPER BELLOWS FILLED WITH WATER
(C) GAUGE FOR CHAMBER AIR PRESSURE
(D) MANOMETER FOR EXCESS PRESSURE IN BELLOWS
(E) STRAIN GAUGE
(F) VENT OR VACUUM CONNECTION
(G) SCREW FOR APPLYING PRESSURE
(H) SCALE FOR LOADS ON SAMPLE
(I) GLASS CYLINDERS
(J) SHAFT FOR SUPPORTING BELLOWS AND SAMPLE
(K) FLEXIBLE CONNECTIONS
FIG. 1-B (NO SCALE)
STRAIN GAUGE STEM
BINDING
RUBBER COVERING
SAND SAMPLE
POROUS
VENT
P
FIG. 1-C
DETAIL OF SAND SAMPLE.
RUBBER COVERING
CLAY SAMPLE
FILTER PAPER
HYDROSTATIC PRESSURE
METAL BASE
SEAL
GASKET
DRAIN VALVE
BOTTOM OF CHAMBER IN WHICH CYLINDERS WERE CONSOLIDATED.
FIG. 1-D
ARRANGEMENT FOR THE CONSOLIDATION OF CLAY CYLINDERS.
LEGEND
No. | DESCRIPTION | MATERIAL
34 | BOLTS | STEEL
33 | GASKET | RUBBER
32 | CYLINDERS | GLASS
31 | COLLAR PLATE | STEEL
30 | UPPER PRES. CHAMBER PLATE | "
29 | DIAL ADJUSTMENT SCREW | "
28 | BEARING FRAME (FOR EXTENSOMETER) | "
27 | EXTENSOMETER (STRAIN GAUGE) |
26 | PLATE | STEEL
25 | PRESSURE HANDWHEEL | "
24 | PRESSURE SCREW | "
23 | BALL BEARING | STEEL
22 | GUIDE POSTS | "
21 | GUIDE | "
20 | BUSHING | "
19 | OIL SEAL |
18 | MAIN LOADING SHAFT | STEEL
17 | MANOMETER VERNIER |
16 | MANOMETER: FOR TEST LOAD |
15 | TUBE: CHAMBER TO MANOMETER | COPPER
14 | MERCURY BULB |
13 | TUBE: BELLOWS TO MANOMETER | COPPER
12 | COUPLING |
11 | CYLINDER | STEEL
10 | MAIN PRES CHAMBER PLATE | "
9 | BEARING PLATES | "
8 | PRESSURE BELLOWS | COPPER
7 | VENT OR VACUUM TUBE | "
6 | ADJUSTMENT SCREWS | STEEL
5 | SAMPLE SUPPORT | "
4 | SAMPLE CAGE | "
3 | SAMPLE BASE | BRONZE
2 | SAMPLE CAP | "
1 | SAMPLE |
APPARATUS FOR CYLINDER TESTS UNDER LATERAL PRESSURE
SCALE

FIG. 1

paper and tied to the end-pieces by winding a strip of thin rubber around the paper which extended over the end-pieces. Thus the failure planes and deformations under test loads were forced into the middle of the cylinder and away from effects of unequal stress distribution at the ends. If the test was to be run under lateral pressure, the rubber covering was doubled, each layer being carefully cemented and bound to the base to prevent any leakage. The drain valve in the base was always open to the atmosphere. The test loads were applied in the direction of the axis of the cylinder and were increased at a uniform rate until failure occurred. Lateral pressure was obtained by air pressure on the outside of the rubber covering. The devices measuring the forces and the deformations were both built into the pressure chamber to eliminate any errors due to friction in the stuffing boxes. As is shown in Fig. 1 the axial deviator force is measured by a mercury manometer connected to the pressure chamber and the copper bellows which support the sample. The principle of the use of bellows was adapted from a testing machine designed by Dr. Gilboy.

As the tests confirm Terzaghi's investigations [(9)],* showing that pressure has no effect on the strength of the clay if no further consolidation is allowed during the test, most of the samples were tested without lateral pressure, as this permitted the use of simpler apparatus. Rubber coverings were always used to prevent evaporation. Consolidation pressures ranged from 0.5 kg./cm.$^2$ to 4.4 kg./cm.$^2$ The results are shown in Figs. 2 to 6 and are summarized in Fig. 6.

In order to facilitate the comparison of the results with those obtained by other types of shearing tests, the results are given in terms of the principal shearing stress, $s$, the magnitude of which is equal to half of the difference between the two principal stresses (Appendix B). Thus $s = P \div 2A$ which is equal to one-half of the deviator compressive stress. As the area of the cylinder changes during the test the stress has been referred to the mean area. Assuming no volume change the formula thus becomes $s = P\ (1-e) \div 2\ A_o$, where $e$ is the axial strain.

In addition to cylinders moulded in the soft state, one set of specimens was formed of material which had been previously consolidated to 1.58 atm. Water was added to some of these samples to get a variation in consistency and the cylinders consolidated to 2.3 atm. Results of these tests showed clearly the paramount importance of the initial consistency at which the samples were moulded. The further study of this effect was done by direct shearing tests which will be discussed later.

* Numbers in brackets refer to Bibliography at end of report.

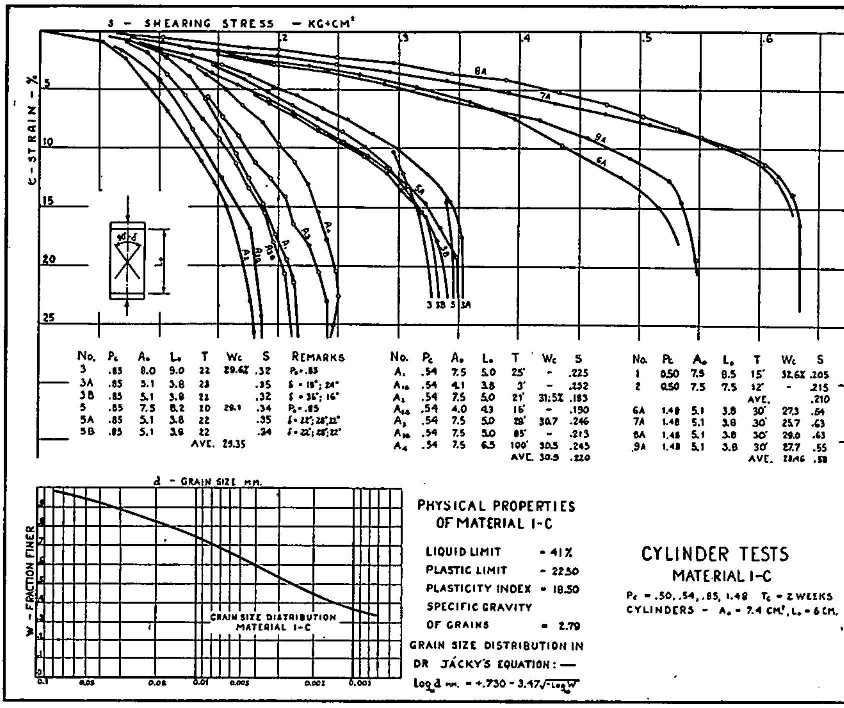

S - SHEARING STRESS - KG÷CM²
ε - STRAIN - %
No. Pc A0 L0 T Wc S REMARKS
3 .85 8.0 9.0 22 29.6% .32 Pc=.85
3A .85 5.1 3.8 23 .35 δ = 18°; 24°
3B .85 5.1 3.8 22 .32 δ = 36°; 16°
5 .85 7.5 8.2 20 29.1 .34 Pc=.85
5A .85 5.1 3.8 22 .35 δ = 22°; 28°; 22°
5B .85 5.1 3.8 22 .34 δ = 22°; 28°; 22°
AVE. 29.35
No. Pc A0 L0 T Wc S
A1 .54 7.5 5.0 25' - .225
A1a .54 4.1 3.8 3' - .232
A2 .54 7.5 5.0 21' 31.5% .183
A2a .54 4.0 4.3 16' - .190
A3 .54 7.5 5.0 28' 30.7 .246
A3a .54 7.5 5.0 85' - .213
A4 .54 7.5 6.5 100' 30.5 .245
AVE. 30.5 .220
No. Pc A0 L0 T Wc S
1 0.50 7.5 8.5 15' 32.6% .205
2 0.50 7.5 7.5 12' - .215
AVE. .210
6A 1.48 5.1 3.8 30' 27.3 .64
7A 1.48 5.1 3.8 30' 25.7 .63
8A 1.48 5.1 3.8 30' 29.0 .63
9A 1.48 5.1 3.8 30' 27.7 .55
AVE. 28.46 .58
d - GRAIN SIZE MM.
W - FRACTION FINER
GRAIN SIZE DISTRIBUTION MATERIAL I-C
PHYSICAL PROPERTIES OF MATERIAL I-C
LIQUID LIMIT = 41%
PLASTIC LIMIT = 22.50
PLASTICITY INDEX = 18.50
SPECIFIC GRAVITY OF GRAINS = 2.79
GRAIN SIZE DISTRIBUTION IN DR JACKY'S EQUATION: —
Log d mm. = +.730 - 3.47√-Log W
CYLINDER TESTS
MATERIAL I-C
Pc = .50, .54, .85, 1.48 Tc = 2 WEEKS
CYLINDERS - A0 = 7.4 CM², L0 = 6 CM.

FIG. 2

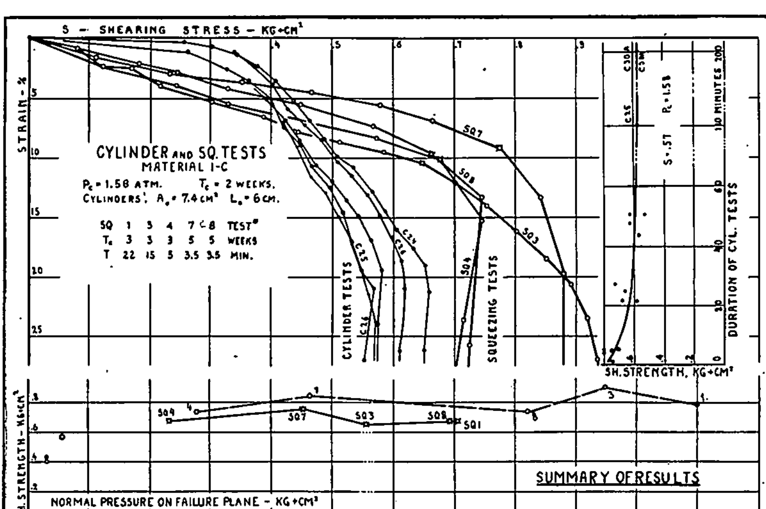

S - SHEARING STRESS - KG÷CM²
STRAIN - %
CYLINDER AND SQ. TESTS
MATERIAL I-C
Pc = 1.58 ATM. Tc = 2 WEEKS.
CYLINDERS: A0 = 7.4 CM² L0 = 6 CM.
SQ 1 3 4 7 & 8 TEST#
Tc 3 3 3 5 5 WEEKS
T 22 15 5 3.5 3.5 MIN.
CYLINDER TESTS
SQUEEZING TESTS
S = .57 Pc = 1.58
DURATION OF CYL. TESTS
SH. STRENGTH, KG÷CM²
SH. STRENGTH - KG÷CM²
SUMMARY OF RESULTS
NORMAL PRESSURE ON FAILURE PLANE - KG÷CM²

FIG. 3

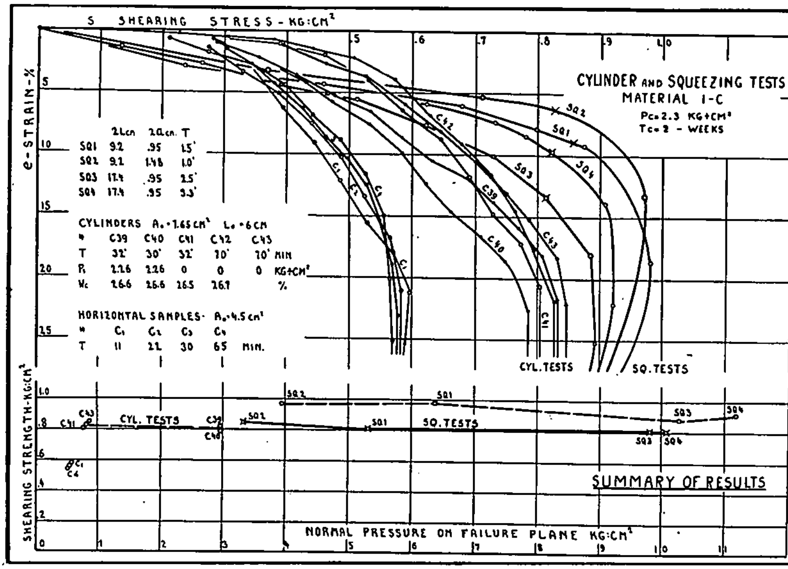

S SHEARING STRESS - KG:CM²
e - STRAIN - %
CYLINDER AND SQUEEZING TESTS
MATERIAL I-C
Pc = 2.3 KG:CM²
Tc = 2 - WEEKS
CYLINDERS
HORIZONTAL SAMPLES
CYL. TESTS
SQ. TESTS
SHEARING STRENGTH - KG:CM²
SUMMARY OF RESULTS
NORMAL PRESSURE ON FAILURE PLANE KG:CM²

FIG. 4

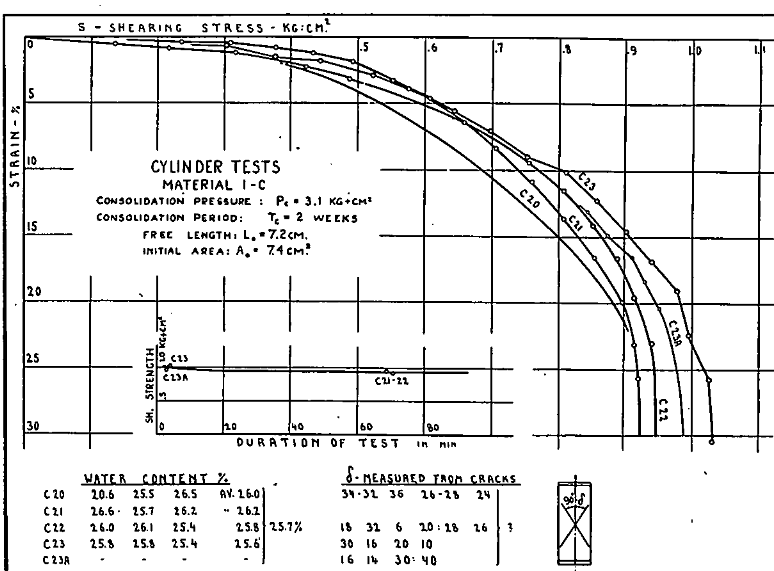

S - SHEARING STRESS - KG:CM²
STRAIN - %
CYLINDER TESTS
MATERIAL I-C
CONSOLIDATION PRESSURE : Pc = 3.1 KG:CM²
CONSOLIDATION PERIOD: Tc = 2 WEEKS
FREE LENGTH: Lo = 7.2 CM.
INITIAL AREA: Ao = 7.4 CM²
SH. STRENGTH KG:CM²
DURATION OF TEST IN MIN
WATER CONTENT %
C20 20.6 25.5 26.5 AV. 26.0
C21 26.6 25.7 26.2 26.2
C22 26.0 26.1 25.4 25.8 25.7%
C23 25.8 25.8 25.4 25.6
C23A - - - -
δ - MEASURED FROM CRACKS

FIG. 5

As was mentioned above, the cylinder tests showed that the lateral pressure had no effect on the shearing strength if no drainage was allowed. Because this point is of considerable interest, and has been the basis of some controversy, it was very essential to confirm the results by some other method. As the direct shearing test is not well suited to this purpose, the author has attempted to obtain a comparison by the use of a novel test which is based on the principles of Rheology and is discussed below.

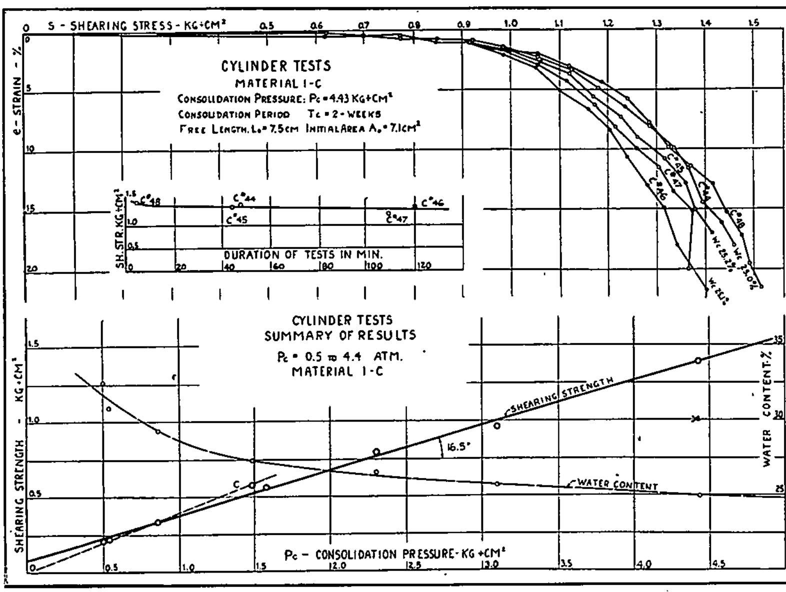



FIG. 6

## *Squeezing Tests*

This test is based on Hencky's principle of equilibrium of materials in the plastic state [(2); (4), page 221]. A thin sample of clay is compressed between two rigid plates until it fails by flowing out on two opposite open sides. The two other sides are sealed to prevent any flow in that direction. To avoid confusion with other types of compression tests, this test is referred to as the squeezing test.

The formulæ for stresses are given in Fig. 7 and are derived in Appendix A. As is seen, the normal stresses vary directly as the length $2L$ and inversely as the thickness $2a$. Thus by doubling the length or halving the thickness we can double the normal pressures on the failure planes. As there is no flow across the vertical center line, we can achieve the same result by closing one outlet and allowing the clay to flow out on one side only ($L_1 = 2L$).

We thus see that the normal stresses can be varied by very simple means. If desired, the materials can be tested under high pressures, as it is not difficult to retard the consolidation during the test. The com-

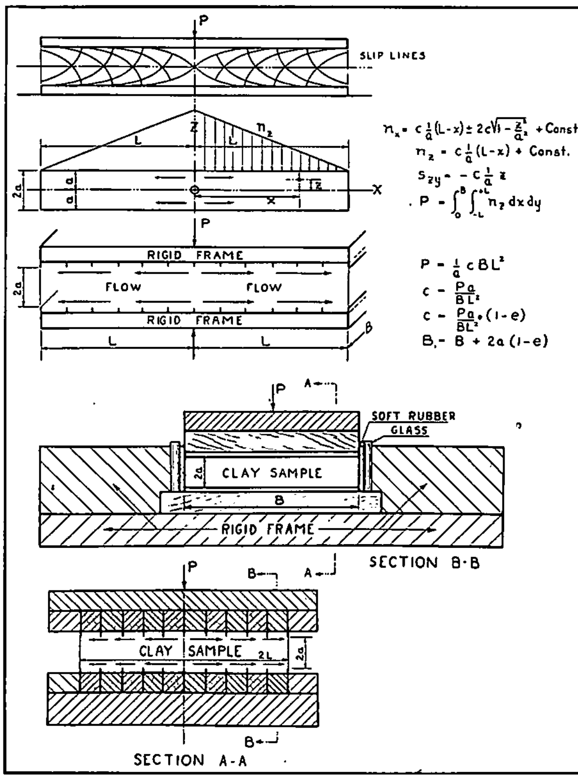



Fig. 7

pressing surfaces must be impervious and so constructed that the material cannot slide at the boundary without developing its full shearing resistance. Unless otherwise noted, the tests described here were performed between the plates shown in the figure. The teeth are of 0.2 mm. spring bronze at a 9 mm. spacing, and project 1.5 mm. into the sample to develop the shearing strength of the material and to force the failure plane deeper into the sample, where the structure is undisturbed and where the drainage is further retarded by the layer of material between the teeth. The spacers are of dense maple thoroughly impregnated to insure watertightness. A better construction would be to set the teeth in milled grooves in metal. To achieve a purely two-dimensional flow, no material should be allowed to escape at the sides.

As is shown in Fig. 7, the bottom and sides of the device are both rigidly connected to a heavy steel plate. Only the top plate is movable. In order to prevent excessive friction between the top plate and sides, the latter are lined with glass and a sheet of soft rubber is placed between the glass and the top plate to prevent escape of clay. Water serves as a lubricant between the rubber and the glass. Friction and drainage along the sides require careful consideration in this test, as the normal stresses are very high. To allow for their effect, the width $B$ in the formula used for computing the stresses has been increased to $B_1 = B + 2a$. It would have been better to correct for this effect by making tests on specimens of different widths. Since the total resistance of the specimen is proportional to the width, and the friction along the sides is independent of width, the effect of this side friction could be eliminated from results of tests that had different widths but were identical otherwise.

The consolidation of samples prior to the test was done by the same method as was described for cylinder tests. A flat sample of clay wrapped in filter paper was mounted on a pervious base and covered with a skin of thin rubber which was sealed to the base. Water pressure was then applied on the rubber and maintained for two weeks to allow complete consolidation. Before being inserted into the testing device, the samples were trimmed to size with a fine wire saw.

Figs. 3 and 4 show the results of the tests and the comparison with results of other tests on material 1C. The highest shearing strength of the materials tested was 2.7 kg. per cm.$^2$, and the highest mean normal pressure, 16 kg./cm.$^2$ (Fig. 11). As the theory of this test assumes Coulomb's Condition of Plasticity, $c$ = constant (Appendix B), it would not apply to materials whose shearing strength depends on normal pressures. Thus, if the clay is allowed to consolidate during the test, a failure never occurs, as the plastic state is never reached.

### *Direct Shearing Tests*

As has been mentioned, these tests were performed to study the effect of initial consistency and the effect of drainage during the test, since they can be performed in shorter time than cylinder tests. The maximum length of path which a water particle has to travel to reach the surface was 16 mm. in the cylinders and could be reduced to 1 mm. in direct shearing tests. As consolidation periods are proportional to the squares of these distances, the time saving is evident.

Fig. 8 shows the principle of the test. The sample is held between two rough surfaces under a vertical force, $V$, and is sheared by a horizontal force, $H$. The vertical force was always allowed to act for a long enough period to fully consolidate the material to the given pressure before the shearing test was started.

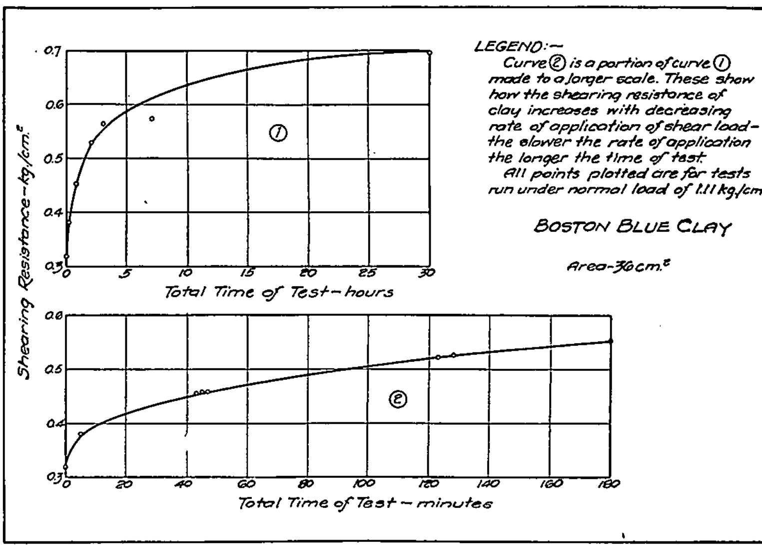



FIG. C8

The tests were made in the shearing machine designed by A. Casagrande [(7)]. Since these investigations are a continuation of the investigations by A. Casagrande and S. G. Albert [(3)], certain of their results and conclusions in regard to clays are of interest. They have definitely established the importance which the time-rate of shear-load application has upon the results of shearing tests. For Boston Blue Clay they have found the relationship shown in Fig. C8, which is from the above-mentioned report. They have also observed that "depending on the previous treatment of the clay, the shearing resistance at the same water content may be considerably different."

By observations of the volume of the sample during the test by means of a vertically mounted extensometer, Casagrande has shown for the first time that fine-grained soils are undergoing additional consolidation during the shearing process; and that, in the case of clays, if this additional consolidation is not allowed to take place the shearing resistance will vary accordingly.

Casagrande has called all angles, correlating shear and normal loads, the "relative angles of internal friction," excepting the largest angles corresponding to the condition where none of the stresses are

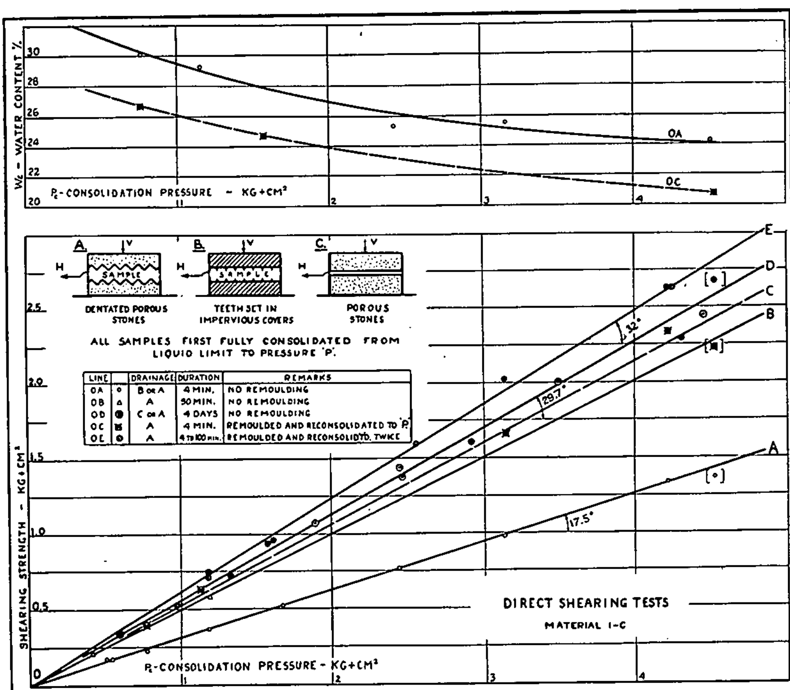



FIG. 8

carried by the water in the soil. The word "relative" is omitted in this report, and the term "angle of internal friction" is used to include all values, since the low angles are of more practical importance, as will be shown later.

Three different types of surfaces used in these tests are shown in Fig. 8. In type A the clay is held between two porous dentated stones made of material used in medium grade grinding wheels. The surfaces in type B are impervious and have teeth of 0.2 mm. spring bronze projecting 2.5 mm. into the clay at 8 mm. spacing. The stones in type C are similar to those in A, but without teeth. In types A and C the longest drainage path is one-half the thickness of the sample; in type B it is 30 mm. It was found that the drainage during the test could be sufficiently reduced by running the tests rapidly, in which case type A surfaces gave the same results as type B. The latter type was used for tests on natural materials and on other samples which were not consolidated in the device itself. Type C was used in tests where it was desired to accelerate drainage and consolidation during the test. The thickness of the sample was reduced to 2 mm. for the same reason. The roughness of the stones was evidently sufficient to develop the full shearing resistance of the clay, as was shown by check tests, using dentated stones.

In all tests described below the material was allowed to fully consolidate to the given pressure before the test proper was started.

In tests represented by line OA in Fig. 8 the consolidation was started at the liquid limit. No further consolidation was allowed during the tests, and the durations were about four minutes.

Line OB represents the tests that were started similarly at the liquid limit, but were made between dentated stone surfaces and were brought to failure at a rate which allowed 90 to 100 minutes for the total duration of the test. Only a few points were determined, as this procedure had been investigated previously by A. Casagrande.

Line OC shows the results of tests that were started at the liquid limit, consolidated to a given pressure, remoulded at the existing water content, and reconsolidated to the same pressure. The tests were performed rapidly, allowing only four minutes for total duration.

Line OE shows the results of tests made after the material of tests OC was remoulded and reconsolidated to the same pressure for the second time. Any further repetition of the same procedure did not further alter the test results. In this state the results are no longer dependent upon the time rate, a four-minute test giving the same result as a 100-minute test.

Line OD shows the results of tests on samples that were consolidated from the liquid limit, and were tested by applying the horizontal pull very slowly. To further help the drainage during the test, the tests marked by heavy points were made with stones of type C, Fig. 8.

All the above tests were performed at room temperature which was maintained fairly uniform at 21° C. by thermostatic control during the winter. As the room could not be cooled in summer the temperature rose and markedly influenced the test results. Points representing a few such results are shown in Fig. 8 in brackets.

As is shown by the results of these tests, the shearing resistance of the material at a given pressure may vary from $0.3p_c$ to $0.63\ p_c$, depending on how the test is made. The deciding factor in this variation appears to be the internal structure of the material, as will be discussed below.

## Discussion of Test Results and Deductions

It would appear to the author that with Terzaghi's principle of consolidation as a basis the behavior of clays in a shearing test can be explained by a simple comparison with the behavior of sands.

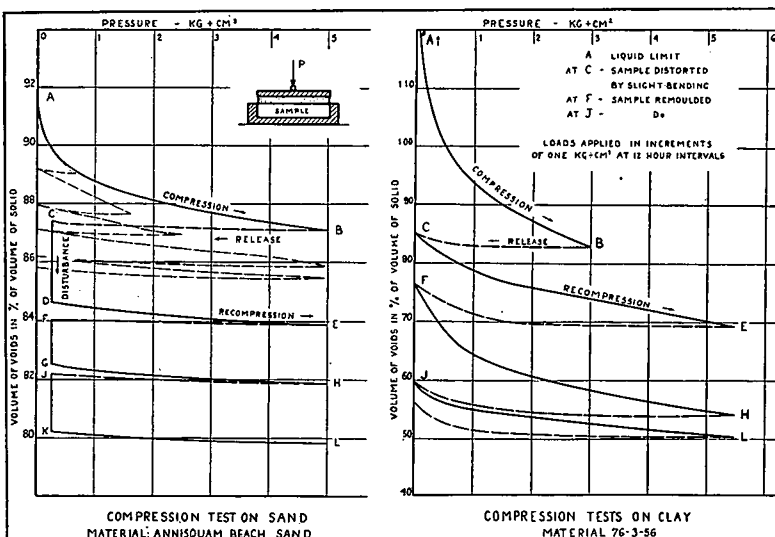



Fig. 9A Fig. 9B

To discuss the properties of clay it may therefore be well to review briefly the behavior of sands under pressure. Fig. 9A shows the results of compression tests on a typical beach sand. When compressed in loose state the material followed the curve AB and rebounded to point C on release. At point C the material was disturbed by a slight hammer blow on the bottom of the container. A considerable deformation, represented by volume change CD, took place before the equilibrium of the structure which was disturbed by the blow was regained. This procedure was repeated as shown in the figure.

We thus see that if pressure alone is applied to a very loose sand the material will follow the curve AB, which therefore represents the density corresponding to the loosest possible state under the given pressure. In order to get a denser structure there must be intermediate disturbances which will destroy the established equilibrium in the internal grain structure. Professor Terzaghi has used the term "conservative structure" to denote this property in sands.

As is seen from Fig. 9B, clay under compression shows essentially the same behavior as the sand just described. A remoulding of the clay is equivalent to the effect of a blow in the case of the sand. Once the internal structure is disturbed a considerable rearrangement of grains must take place before equilibrium is restored and the internal structure again becomes stable. In clays the phenomenon is complicated only by cohesion and the time element of the consolidation process, as the internal structure cannot fully readjust itself and acquire a denser state until some of the water is squeezed out of the voids.

The structure of clay is a skeleton of fine mineral particles with all interstices filled with water. As in sands, this skeleton may be loose or dense, depending on the previous history of the material and not on the existing normal pressure alone. When a disturbance occurs, a part of the skeleton collapses, but immediate compaction is prevented by the presence of entrapped water which now must carry part of the load, as in the cylinder of a hydraulic press. Depending upon the permeability of the clay and upon the length of drainage path, a considerable period of time may be required before the excess water escapes and equilibrium is again restored. During this period the load is gradually transferred from the water back to the grains.

Therefore, a clay consolidated from the liquid limit can offer only the shearing resistance corresponding to its loosest structure at the given consolidation pressure, provided no further compacting is allowed during the test. Only if the material was disturbed at an intermediate stage of the consolidation process can it offer a resistance corresponding to a denser and more stable internal structure. The same effect could be obtained by allowing the material to compact and readjust its grains during the process of the shearing test.

In the tests represented by the line OA, Fig. 8, the drainage, and consequently any further consolidation during the test, was prevented. The material could therefore offer only what shearing strength it had prior to the test. As the test was started at the liquid limit, the structure before testing was close to the loosest possible under the given consolidation pressure, which resulted in a relatively low shearing resistance. When the same material was remoulded and reconsolidated to the same pressure the resultant structure was much denser and more stable, and offered a considerably higher resistance to shearing, as is shown by the tests represented by line OC. As the structure was further compacted by the second remoulding, the results were correspondingly higher (line OE). The limit of strength is reached when the material is reduced to the densest state possible under the given pressure. Any repetition of the process then has no further effect on the structure and on the shearing strength. Absence of further compacting is indicated, also, by the fact that the resistance in this state is no longer dependent upon the time rate, which is an important factor in the loose state. The OA, OB and OD samples were all identical when the shearing tests were started. As the samples were in the loose state, they were all capable of further compaction when distorted under pressure. The longer the duration of the test the more time did the material have for drainage and for readjustment of its structure, and the higher was the resultant resistance. The highest value observed was 0.574 $p$ in a test of 48 hours' duration, which is somewhat lower than the resistance of a sample that was in a dense state at the beginning of the test (line OE). It is of interest to note that a similar difference occurs in sands, as is seen from comparison of tests started in the loose and in the dense state. Evidently the failure occurs before the action of distortions caused in testing fully readjusts the structure. If the duration of tests is shorter, the readjustment is only partial, as is confirmed by the intermediate values obtained in tests of 100 minutes' duration (line OB).

It is evident that the density or water content alone cannot determine the strength of the material, as for a given density the internal structure can differ widely. If the structure is thought of as consisting of small columns of grains, its strength would depend largely upon the efficiency of bracing. In determining the strength of the framework the density is thus of secondary importance as compared to the efficiency of the arrangement of grains. A remoulding will, without any change in density, destroy the efficiency of the arrangement to the extent that only a fraction of the former strength remains. Consolidation following the remoulding leads to further compacting and to a new stable arrangement. Due to greater density the columns are now shorter and more heavily braced, with the result that the material can offer a higher resistance.

The importance of initial density is thus evident. For a given disturbance, the denser the initial structure the less is the quantity of excess water, and the easier is it for the grain structure to regain equilibrium.

It is of interest to note in this connection that Dr. Jaky of Budapest has made a mathematical derivation for the angle of internal friction of sand as a function of density of structure. His results for quartz give $\phi = 10°$ for the loosest, and $\phi = 40°$ for the densest state [(10)].

The same considerations apply to the results of the cylinder tests described earlier and shown in Figs. 2 to 6. The interpretation given to the results is shown by the heavy line in Fig. 6. As the cylinders were not moulded in the liquid state the line is not drawn through the zero point. The effect of moulding at a consistency slightly denser than the liquid limit is more pronounced in samples consolidated to low pressures where the final $W_c$ is closer to the initial. The conditions for these tests are therefore somewhat similar to those of OC tests (Fig. 8), discussed earlier. It would hence appear unreasonable to interpret the results as shown by line OC in Fig. 6.

As is seen, the shearing strength of material 1C, consolidated from the liquid limit without intervening disturbances, increases with the pressure at a rate corresponding to an effective angle of internal friction for relatively impervious materials of 16.5 degrees. One kg. per cm.$^2$ of consolidation pressure increases the shearing strength by 0.296 kg./cm.$^2$

The shearing resistance observed in cylinder tests is somewhat lower than was obtained in direct shearing tests (line OA, Fig. 8). This is probably caused by the effect of viscous resistance at the rapid rates used in the latter case. As is seen from the figures, the strength of the cylinders was higher when the tests were run rapidly, evidently for the same reason. To eliminate this effect, the time rates of cylinder tests were chosen to give durations of about 100 minutes. The durations in direct shearing tests of series OA could not be increased on account of difficulty in preventing drainage.

In sands or other pervious materials readjustments to imposed stresses can take place rapidly. After the collapse of a given grain structure the material will compact, even under its own weight, and rapidly restore a new and more stable equilibrium at a denser state. As all normal stresses are thereby carried by solid mineral from grain to grain, they develop friction and contribute to the shearing resistance of the material. Sand therefore follows Mohr's Condition of Plasticity (Appendix B) [(5), (4), pages 45, 61]. Conversely, in clays the normal stresses after a collapse are partly carried by water trapped between the solid grains, and do not develop the full friction. Hence, in a rapid test with no readjustment allowed during the test they follow Coulomb's Condition of Plasticity [(4), pages 60, 184].

The material was fully consolidated prior to all tests, and consequently had acquired a shearing strength corresponding to the consolidation pressure at the given structure. As further consolidation of the cylinders was prevented during the test, the material was unable to readjust its structure to the excess stresses imposed during the test. These stresses were therefore unable to contribute to the strength. Consequently, the shearing resistance of the material was not affected by the magnitudes of normal stresses applied by air pressure.

It should be noted that the tests under high pressures were all performed on samples that were moulded at the liquid limit and consequently had a relatively loose structure.

It is probable that the shearing strength of a clay in a relatively dense state would be affected by changes in normal pressure, even if no drainage is allowed. As such conditions are not likely to occur in natural deposits, and the question is consequently of minor practical importance, such tests have been deferred. However, this point might explain the results obtained by investigators who state that the shearing strength of a clay increases with the normal stresses, even when drainage is prevented.

During the process of the shearing test the initial stability of the grains is disturbed by distortions caused by external loads. This disturbance has an effect similar to partial remoulding. If drainage were allowed, and the test run at a sufficiently slow rate to allow a complete readjustment of internal structure to each increment of the imposed stresses before the next increment is added, a clay would behave like a pervious sand. It would follow Mohr's Condition and develop a high shearing resistance, even if the test was started at the liquid limit, as was shown by the tests OD in Fig. 8. The effect of this process in a cylinder test would be that the shearing strength along various planes would be no longer uniform (Appendix B). Those planes which are under higher normal stresses will develop a correspondingly higher resistance. As the result, the least favorable planes would be steeper than the planes of principal shearing stresses by $\frac{1}{2}\,\phi$, and bear the well-known relation of $45+\frac{1}{2}\,\phi$ to the major principal plane. The above conditions are not met in a cylinder test on clay, as confirmed by the fact that the shearing strength is independent of normal pressure. It would appear, therefore, that there is no theoretical or experimental justification for the practice of determining the angle of internal friction of clays from orientations of failure planes on compressed cylinders.

If Coulomb's definition of the angle of internal friction is taken to refer to the ultimate shearing resistance, irrespective of the changes in structure, its value in our material, 1C, would be about 30°, which is of the same order of magnitude as its values for loose sands. The difference between the two materials would thus narrow still further, as under this definition of $\phi$ both materials follow the same law of plasticity. The only remaining difference lies in the time element, which depends on the permeability.

In figuring the normal stress intensities acting on the failure planes in the direct shearing and squeezing tests, the vertical load has been divided by the area of the sample. For the squeezing test this gives the mean pressure, while actually the vertical stresses increase with the distance from the outlet. In the cylinder tests the pressure is referred to the planes of maximum shear, where the normal deviator stress equals $s$ in magnitude. This value has been added to the lateral pressure to give the total normal pressure on the failure plane. The highest pressure used in material 1C was 4 kg./cm.$^2$ in cylinder tests and 12 kg./cm.$^2$ in squeezing tests. The shearing resistance obtained by squeezing tests is higher than was obtained in cylinder and direct shearing tests. It is evident, however, that this increase is not due to higher pressure, as is seen from comparison of squeezing tests performed under different pressures.

In the cylinder and direct shearing tests the failure can occur along a single sliding surface, while in the squeezing test the flow must occur simultaneously along an infinite number of failure surfaces. The squeezing test therefore measures the average strength of the sample rather than the strength of the weakest plane, and, unless the material is perfectly uniform, should show a higher ultimate resistance than the other tests. The point of ultimate resistance in the direct shearing and cylinder tests would thus correspond to the point where the stress-strain curve begins to break in the squeezing test. The distance between this point and the ultimate resistance should be smaller in homogeneous materials. Both of these values are shown in the diagrams, and the estimated breaking points are indicated by crosses.

## Shearing Tests on Natural Samples

For the investigation on Boston Blue Clay, the samples tested were selected to represent the extreme varieties of the material available. The first materials 76–4–50 and 76–3–60A were rather fat and very uniform; while the samples 76–2–56 and 76–3–54 showed sharp variations in their fine structure. Both materials came from Boston Harbor from drill holes 15 to 18 meters (50 to 60 feet) below the surface. Cylinders and samples used in squeezing tests were cut with a fine wire saw and consolidated under hydrostatic pressure in the same way as the 1C samples previously described.

When cutting cylinders a device was used where the sample could be turned around its axis. About twenty cuts made parallel to the axis with a taut guided wire gave specimens that were very nearly cylindrical.

Figs. 10, 11 and 12 show the results of the tests.* As could be

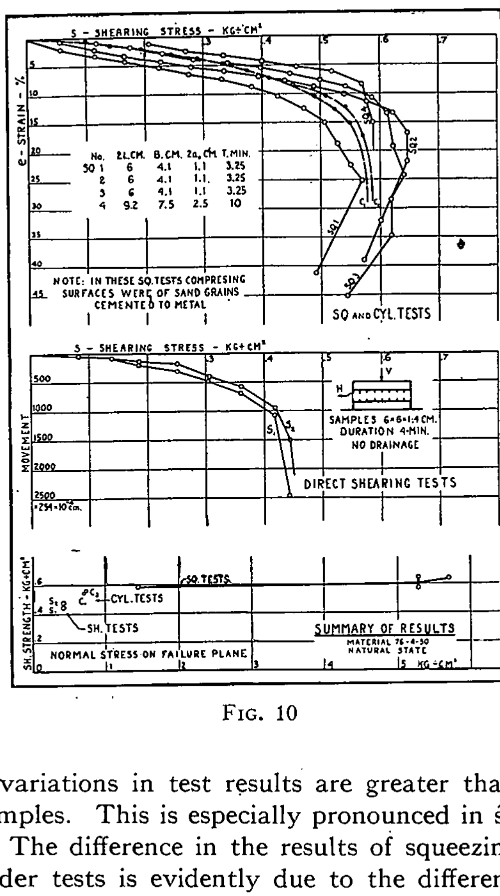
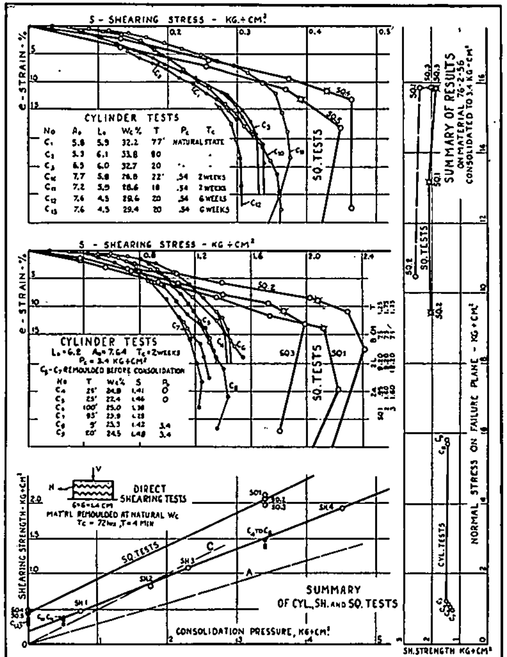



FIG. 10

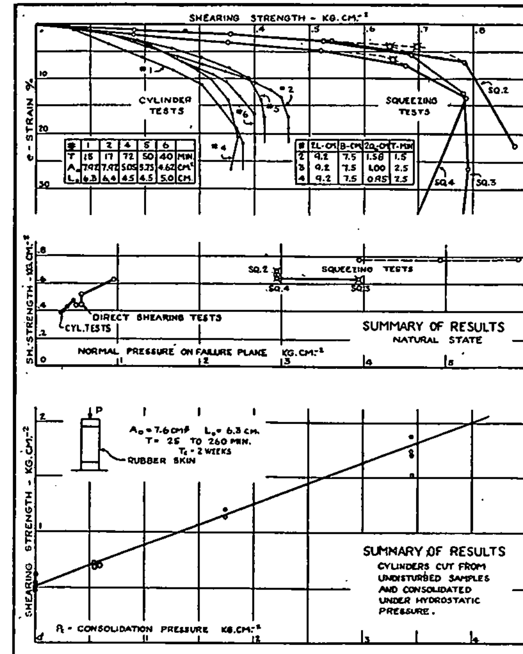
expected, the variations in test results are greater than in the homogeneous 1C samples. This is especially pronounced in samples 76–3–54 and 76–2–56. The difference in the results of squeezing tests as compared to cylinder tests is evidently due to the difference between the average strength and that along the weakest plane. The existence of non-uniformity is evident from photographs of the material in semi-dry condition (Fig. 13), which show white streaks of fine sandy material

* Results on material 76–3–60A are shown in Fig. 7 of the other report appearing in this issue.



SHEARING TESTS ON SAMPLE 76-2-56

FIG. 11



SHEARING TESTS ON SAMPLE 76-3-54.

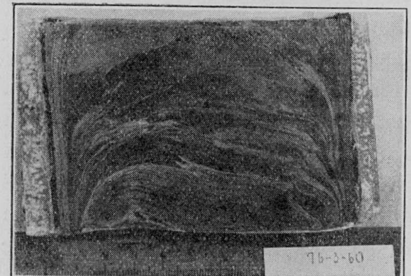
FIG. 12

between layers of fat clay. Since the results of squeezing tests are constant and independent of the normal pressure, it would appear that the higher resistance was not caused by further consolidation during the test. Cylinders C8 and C9 (Fig. 11), tested under lateral pressure, gave the same results as the cylinders tested in the open.

Some of the cylinders 76–2–52 were remoulded before being consolidated to higher pressures. As no difference was noticed in the results, the disturbances caused by sampling and handling appear to have been equivalent in this respect to complete remoulding. The results of direct shearing tests on remoulded samples gave very nearly the same results as the cylinder tests (Fig. 11). The slightly higher values obtained in the direct shearing tests are probably caused by a higher viscous resistance at faster rates.

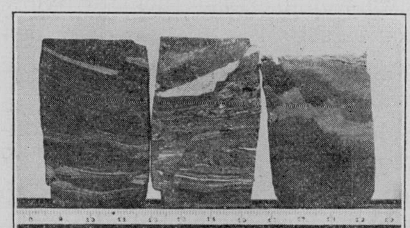
In nature most clays are deposited with a very loose structure. As they are subsequently put under the burden of gradually increasing deposits, and are slowly consolidated, we would expect them to follow a compression curve corresponding to curve AB (Fig. 9B), as described earlier. The structure would therefore be close to the loosest possible for the given pressure. Such a material would be very sensitive to a disturbance, inasmuch as a collapse of the structure requires that considerable deformation occur before the equilibrium is restored at a denser consistency. When samples are taken from drill holes according to present practice, a piece of 5-inch pipe is driven into the bottom of the hole, which may cause a considerable disturbance in the structure of the clay, as can often be seen from the bending of the strata. The stratification can be seen best when the clay is partially dried in a very humid air, when portions having different hygroscopic properties show a marked variation in color. Photograph A (Fig. 13) shows a sample of clay that has been considerably disturbed. Another factor, the effect of which is invisible to the eye, is the removal of the overburden and of the lateral confinement.

The effect of these factors is that the equilibrium established in the internal structure of clay by nature is disturbed. Hence, as compared with the behavior of the material in nature, specimens tested in the laboratory would show higher initial compressibility, lower shearing strength if tested before reconsolidation, and higher shearing resistance if tested in the reconsolidated state. Increased compressibility is caused by the deformations which must occur before a new internal equilibrium is established. To discuss the effect of disturbance on the shearing resistance we will consider the case of a loose material and assume that a partial disturbance is caused at point L (Fig. 14). If now tested before being reconsolidated, the material would show a shearing strength that may be close to PL, or may be only a fraction of that value, depending on the degree of disturbance.

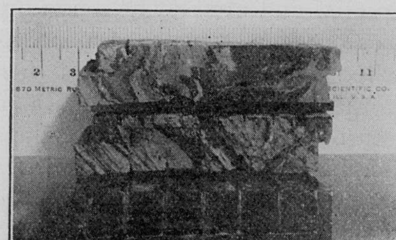


13A. — Vertical Section through Drill Hole (Sample)

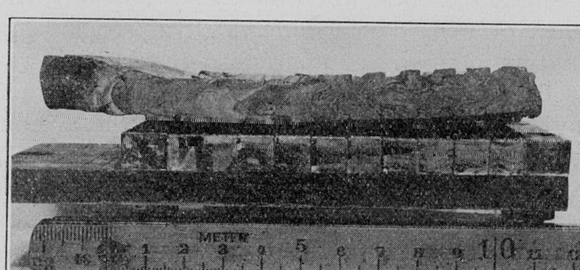
13B. — Clay Cylinders after Testing



13C. — Cross Section of Clay Cylinders after Testing



13E. — Cross Section of Shearing Test Samples



13D. — Cross Section of Squeezing Test Samples

FIG. 13

If samples were now reconsolidated to pressures $OP_1$ and $OP_2$ the shearing strength would be $MP_1$ or $NP_2$. Because of a more stable internal structure, both these values are higher than the shearing strength that the material would have under the same pressures in nature. To illustrate this by the results of tests, we can use the direct shearing tests performed on remoulded material (76–2–56, shown in Fig. 11). The shearing resistance of the sample consolidated to 4.5 kg./cm.$^2$ was 1.95 kg./cm.$^2$, while under the same pressure in nature the shearing strength would not exceed 1.37 kg./cm.$^2$, as is shown by results of tests started at the liquid limit (OA). For reasons which will be discussed later, the resistance in nature could be considerably lower than is obtained in a laboratory test started at the liquid limit.

It is of interest to note that the test data shown in Fig. 14 allow a number of conflicting interpretations. As is seen from the plot of test results, an experimenter might say that the material has a certain "cohesion, $k$," and "angle of internal friction, $\phi$." Their values would, however, depend upon the degree of disturbance and upon the reconsolidation pressures used in the test. The greater the disturbance, the lower will be the value of $k$ and the higher the apparent value of $\phi$. In addition, all results are further influenced by the time factors of consolidation and swelling, which have not always been duly considered. It would appear to the writer that these factors are partly responsible for the complexity of values reported by various investigators.

For the reasons stated above, the author cannot agree with the practice of stating the results of such tests in equations of the type $c = c_1 + p \tan \phi$. Such equations can well represent the test results, but may lead to erroneous conclusions, since they do not represent the properties of the material in nature. As will be shown later, the above applies especially to materials having the property of hardening independent of pressure.

As was seen earlier (line OC, Fig. 8), the shearing resistance of clay that has been remoulded once and reconsolidated to the same pressure is a fairly constant fraction of that pressure. This principle suggests a possibility of an independent method for determination of the consolidation pressure of a given natural deposit from results of shearing tests on remoulded samples. Tests to determine a line such as MN, Fig. 14, may be made on a remoulded sample, and the intersection of this line with line OC, Fig. 8, determined. As undisturbed samples are often not available, such a method might be helpful in practice.

To test this possibility, the line OC in Fig. 11 was determined from results of direct shearing tests on samples that were consolidated from the liquid limit (line OA), remoulded and reconsolidated to the same pressure. The line OC intersects the line representing the tests on natural remoulded samples at $p_c = 1.7$ kg./cm.$^2$, while the actual overburden was about 1.9 kg. per cm.$^2$

The identical procedure was repeated on material 1C which had been previously consolidated to 1.59 kg./cm.[2] The results are shown in Fig. 15. As is seen, the value of $p_c$ determined indirectly from the

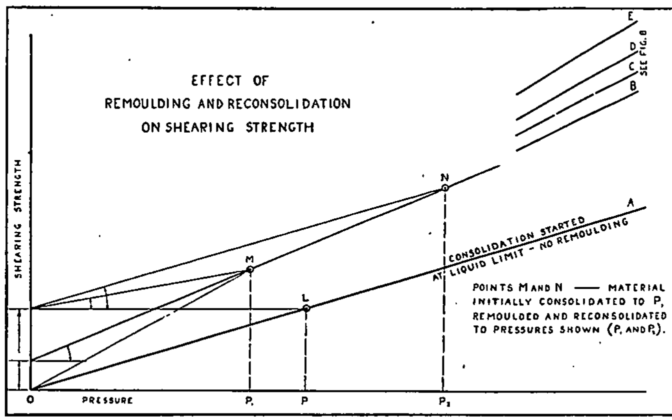



FIG. 14

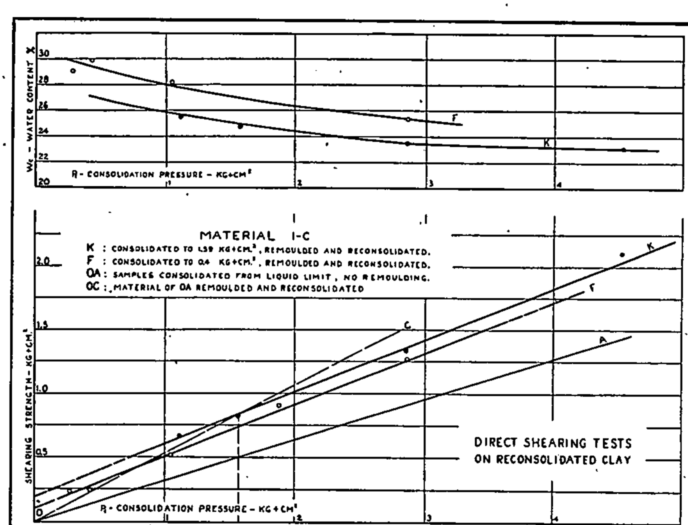



FIG. 15

results of shearing tests agrees well with the true value. The same cannot be said about the samples which were preconsolidated to 0.4 kg./cm.[2], but this pressure is evidently too light and the corresponding consistency too close to the initial to give consistent results.

Some clays have a property of hardening, even when under no external pressure, either mechanical or capillary. Shortly after moulding the material acquires an appearance and feeling similar to that of jelly, and with time develops an appreciable cementation. As such cementation increases the stability of the internal structure, and thus tends to prevent the grains from compacting under the weight of slowly growing deposits, such clays would have a very loose but well-bonded structure with large voids.

As was shown earlier, a very large volume decrease must occur before a loose structure reaches a new equilibrium after a disturbance. Although hard in undisturbed state, the above clays are therefore very much affected by remoulding. In addition to the effect of a large amount of excess water present, the shearing strength is impaired by the loss of cementation.

Clays deposited in the bottom of preglacial Lake Hitchcock at Turners Falls, Mass., and the Laurentian clays show all these properties very markedly. In Boston clays the effect is not so marked. While difficult to measure, it can be seen from the effect of the age of a sample on the orientations of sliding planes. The cracks on the surface of cylinders become very irregular in samples which are allowed to set before testing.

The structure of a clay in nature can therefore be less dense than the structure under the same pressure in the laboratory. Accordingly, the shearing resistance may be below the shearing strength obtainable in the laboratory if the direct effect of cementation on shearing strength is less than the effect of the looser structure caused by cementation.

The effect of cementation in increasing the resistance of a grain structure to normal stresses is similar to the effect of lateral bracing in columns. It would appear to the author that it might be responsible for the extremely loose materials often found at great depths where the normal pressures must be very high. Under static surface loads shearing stresses at such depths are low, and therefore, also, the distortions, which are the important factor in compacting deposits. However, if the equilibrium is impaired by vibrations, such deep lying strata may be responsible for appreciable settlements.

Similar phenomena may occur, also, in clays which do not possess the property of hardening independent of pressure, as explained by A. Casagrande [(3), page 180].

As the process of hardening is markedly affected by temperature, and its total effect on the properties of clays has not been fully investigated, care should be exercised in interpreting the results of laboratory tests on materials having this property. As their behavior involves, also, the laws of Colloidal Physics, an interpretation based solely on ordinary Mechanics may be very misleading.

## Remarks on Testing Procedure

In practice we are interested in the shearing strength when analyzing the safety of foundations and embankments. Whenever possible, the shearing strength should be determined from tests on undisturbed samples. As we are interested in the strength in the natural state, any further consolidation under testing loads should be prevented. This can be done by retarding the drainage and by testing the sample under light normal stresses.

A simple cylinder test under no lateral pressure would therefore appear to the author to be more reliable than the present type of direct shearing test, where the resistance is in addition more liable to be affected by local stress concentrations close to the failure planes.

The squeezing test needs further development and criticism. The test requires very simple apparatus, and the results on materials tested thus far have been very consistent. The practicability of using rigid circular plates with a radial flow has not yet been investigated. The main advantage of the cylindrical type would be the absence of friction on the sides. As was discussed earlier, the cylinder and the direct shearing tests give the resistance of the weakest plane of the test specimen, while the squeezing test gives the average strength of the sample. Which of these values is more significant will obviously depend upon the nature of any given problem and general character of the given material.

If no undisturbed samples are available, the shearing strength of the soil has to be determined indirectly. The pressure to which the material is consolidated is in most cases equal to the weight of the overlying strata. As the shearing resistance for a given pressure will have to be determined by test, the question now may arise as to which type of test to use for this purpose.

In nature clays are deposited in the liquid state, and the strata are usually thick as compared with the dimensions of samples tested in the laboratory. Remembering that consolidation periods are proportional to the squares of drainage distances, we are led to the conclusion that in analyses of stability clay has to be considered as an impervious solid having a constant shearing strength corresponding to its consolidation pressure. It would therefore appear that the procedure in making shearing tests if no undisturbed samples are available should be to fully consolidate the material from the liquid limit to the given pressure, allow no drainage during the shearing test, and reduce the duration of the test to a minimum. It is evident that the rule should not be taken to include cemented materials, clays with very favorable drainage conditions, or relatively pervious materials, where it is often possible to take advantage of consolidation and the resultant increase in strength, as, for instance, in embankments resting on silt deposits.

## Remarks on the Question of Lateral Pressure at Rest

As the question of lateral pressure in nature is of considerable interest, the question arises of the possibility of its determination from results of tests on vertical and horizontal samples. If a clay sample is consolidated in the laboratory under a vertical load, so that the compaction proceeds in the vertical direction only, then horizontal and vertical samples show markedly different properties. Cylinders with axes horizontal show a strength of about 75 per cent of that of cylinders whose axes are vertical. A variation could not be observed in cylinders of natural samples of material No. 76 tested in horizontal, vertical and 45° directions; but since the drill hole samples are partially disturbed the evidence is not conclusive.

It would appear to the author that the results of such tests bear no relation to Poisson's ratio. The planes of principal shearing stresses are in both cases the same, the only difference being the direction of the deformations. The difference in shearing resistance may thus be due solely to non-isotropy caused by consolidation in one direction only.

It would also appear that the skin friction tests with vertically and horizontally imbedded metal cylinders offer no proof, as a rigid body in a compressible medium causes higher stress concentrations in its vicinity. A higher resistance of the horizontal cylinders might thus be merely due to the rigidity of the metal. If the question is considered purely from the standpoint of distribution of stresses under a uniform loading, of infinite extent, according to the Theory of Elasticity one is led to the conclusion that both the vertical and horizontal stresses must be equal.

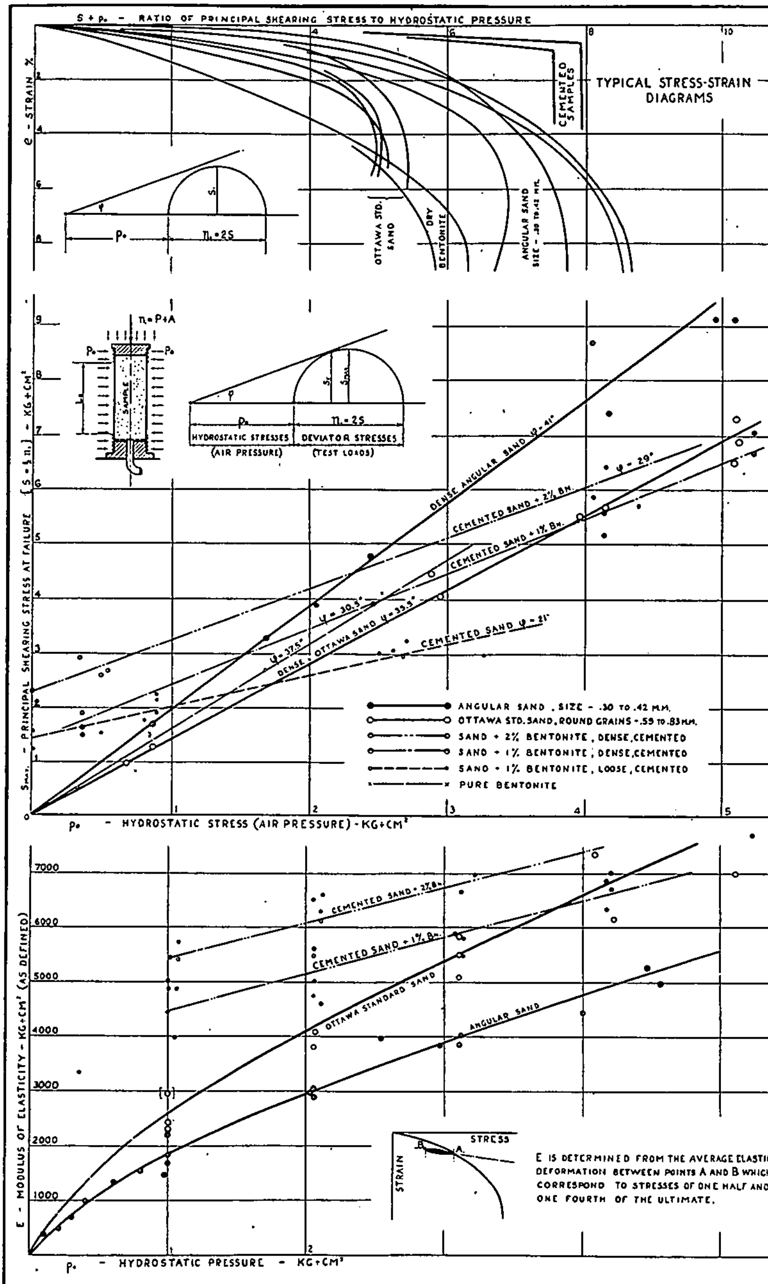
If we use Carothers' solution for a uniformly loaded strip —

$$n_1 = \frac{p}{\pi}(\alpha + \sin\alpha)$$

and

$$n_2 = \frac{p}{\pi}(\alpha - \sin\alpha) \quad [(8), \text{ pages } 58,\ 156]$$

where $\alpha$ is the angle subtended by the loaded area, and apply it to a strip of infinite width, where $\alpha = \pi$, we get $n_1 = n_2 = p$, which is inde-




RESULTS OF CYLINDER TESTS ON SAND

FIG. 16

pendent of the elastic constants of the material. Similar results are obtained for stresses in a material compressed in a rigid box with frictionless vertical walls which allow no horizontal movement and no vertical shear. These boundary conditions are the same as in a uniformly loaded medium of infinite extent. It would also appear that a vertical movement of grains during the compaction of soil does not necessarily mean that the elastic strain in individual grains is mainly vertical.

### Shearing Resistance of Sands

This subject has already been investigated by A. Casagrande [(3)] on the direct shearing machine, and the values obtained compare well with the results of cylinder tests (Fig. 16).

It is of interest to note that Ottawa Sand, consisting of round grains, has a lower resistance but a higher Modulus of Elasticity than angular sand, evidently because the contact surfaces are spherical rather than sharp.

The resistance of cemented sands increases with the pressure at a considerably lower rate than that of uncemented sands. If cemented the particles are no longer free to adjust themselves to meet the stresses, and once the cementation is broken the failure proceeds so suddenly there is no time for readjustment. The failure occurs at $e < 1$ per cent, as shown in Fig. 16.

In the tests represented in the figure, cementation was obtained by mixing bentonite with sand, forming into cylinders and drying before testing. The samples were sealed in a rubber skin and tested under hydrostatic pressure, as described above. Similar tests were performed on the same material with the cementation broken, and on pure bentonite dried and crushed to the size of sand grains. The tests show that both of these materials develop approximately the same angle of internal friction as plain sand.

## Appendix A

*Theory of Squeezing Tests.* — It was shown by the results of side pressure tests that the shearing strength of clay is independent of normal stresses if no consolidation is allowed. The condition of plasticity for this case is $c = \sqrt{s_{xz}^2 + \left(\frac{s_x - s_z}{2}\right)^2}$. For the case of two-dimensional flow of such a material, solutions have been made by Hencky and Prandtl [(2); (4), page 221]. The solution for parallel boundaries applies to the case of a plastic material, squeezed between two rough, rigid plates with freedom to move toward two opposite sides only. It can be used as the theoretical basis for a test for determination of the shearing strength of clay. The resultant flow lines are families of cycloids — $y = a \cos 2\beta$; $x = \mp a\, [2\beta \pm \sin 2\beta + const.]$ (Fig. 7). The stresses are —

$$n_x = \frac{c\,(L-x)}{a} \pm 2c\sqrt{1-\frac{z^2}{a^2}} + const.$$

$$n_z = \frac{c\,(L-x)}{a} + const.$$

$$s_{xz} = -\,c\,\frac{z}{a}$$

At the boundaries $z=a$ and $z=-a$ we have —

$$n_z = n_x = \frac{c}{a}\,(L-x) + const.$$

$$s_{xz} = -\,c$$

Integrating the normal stress over the area of the plate from $-L$ to $+L$ and from zero to $-B$ in the $y$ direction, we get —

$$P = \frac{1}{a}\,c\,L^2\,B; \quad c = \frac{P\,.\,a}{BL^2}$$

As the thickness ($2a$) changes during the test, the following form is convenient:

$$c = \frac{P\,.\,a_o\,(1-e)}{BL^2}$$

where $2a_o$ = initial thickness, and
$e$ = change in thickness $\div 2a_o$.

As these equations are derived from principles of Rheology, and refer to the plastic state, they do not give the true stresses at the beginning of the test when the plastic state has not been reached. Since we are here interested only in the ultimate resistance, this is of no importance.

## Appendix B

The resistance against failure may be analyzed by determining the stresses on all planes through a point in a stressed material and comparing these stresses with the strength on corresponding planes. It is evident that the most dangerous plane will be where the difference between strength and stress is a minimum.

Consider the strength of a sand cylinder under an hydrostatic pressure $p_o$ and subjected to an axial load $P$ which causes a deviator compressive stress $P \div A = n_1$ (Fig. 17). Mohr's circle drawn with a radius equal to the difference between the two principal stresses ($p_o$) and ($p_o + n_1$) helps to visualize the stresses on any plane. The horizontal plane corresponds to point $B$ on the circle. The only stress acting in this plane is $OB = p_o + n_1$. To find the stresses on plane $\alpha$, lay off angle $DAB = \alpha$, or angle $DCB = 2\alpha$, as is shown in the figure. The normal stress on the plane $\alpha$ is $n_\alpha = OB_1 = \frac{2p_o + n_1}{2} + \frac{n_1}{2} \cos 2\alpha$, and the

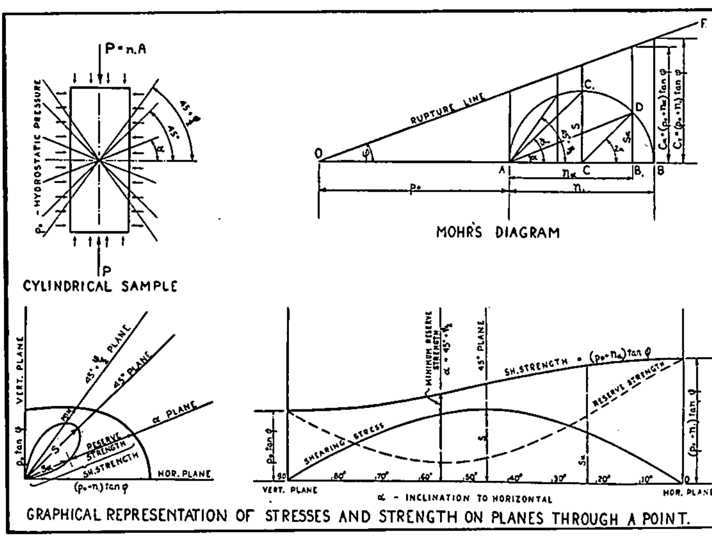
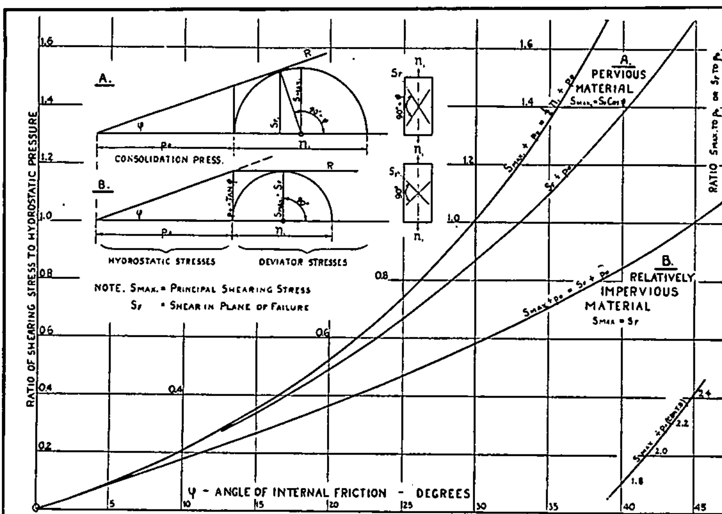



FIG. 17

shear, $s_\alpha = B_1D = \frac{n_1}{2} \sin 2\alpha$. The shearing stress reaches a maximum when $\alpha = 45°$ (Point $C_1$), where it becomes the principal shearing stress, $s = \frac{1}{2} n_1$. The total normal stress on this plane is $p_c + \frac{1}{2} n_1$. The stress conditions on any plane are easily found from Mohr's diagram. To judge where failure is most likely to occur we have to consider the question of strength in the different planes. According to Coulomb's law of friction, the shearing resistance is the normal pressure times $\tan \phi$. On any plane the strength thus is $(p_o + n) \tan \phi$. Its value is thus greatest on the horizontal plane, and gradually drops to a minimum on the vertical, as is shown in the figure. Since the difference between the strength and the stress shows the safety against a failure, it is seen that the least favorable plane is where $\alpha = 45 + \frac{1}{2}\,\phi$. The plastic state begins here when the shearing stress reaches the strength of the material on this plane. In the graphical representation the plastic state begins when the Mohr's circle touches the line OF drawn at the angle $\phi$ to $OB$. $OF$ is called the rupture line, and the above rule is called Mohr's Condition of Plasticity. It is essentially a graphical representation of Coulomb's law of friction ($c = p \tan \phi$) in the case of two-dimensional stress. Expressed in terms of total stresses on any co-ordinate planes,




RELATION BETWEEN ANGLE OF INTERNAL FRICTION AND SHEARING STRENGTH.

FIG. 18

Mohr's Condition is $\frac{n_x + n_y}{2} \sin \phi = \sqrt{\frac{(n_x - n_y)^2}{2} + s_{xy}^2}$ [(4), page 45].

As was seen, the shearing strength in any plane is $(p_o + n_\alpha) \tan \phi$ of which $p_o \tan \phi$ is caused by hydrostatic stresses and $n_\alpha \tan \phi$ is contributed by deviator stresses imposed in testing the sample.

A relatively impervious material requires time to consolidate to pressure $p_o$ and develop the shearing strength $p_o \tan \phi$. Such a material cannot readjust its structure to stresses imposed in a rapid test, and these stresses therefore do not further contribute to the strength. The rupture line becomes parallel to $AB$ and $c = p_o \tan \phi = const.$ (Fig. 18B), which is commonly called Coulomb's Condition of Plasticity. Since this condition is caused by low permeability, it appears more reasonable to call this the case of a relatively impervious material rather than a material that has no friction. When interpreting the test results the angle of friction can be computed from the relation between the principal shearing stress at failure ($s_{max.}$) and the hydrostatic pressure ($p_o$). This relation is shown graphically in Fig. 18 for both materials. If a material has been consolidated to 1 kg./cm. and fails at $s = 0.3$ kg./cm.$^2$ in a rapid test, the ratio $s : p_o$ equals 0.3, which corresponds to 16.8° in an impervious material. At the same $\phi$ the value of $s$ in a pervious material would be .42 $p =$ .42 kg./cm.$^2$

## Acknowledgments

For instruction in Soil Mechanics, the author is indebted to Professor K. Terzaghi, on whose principle the present work is based. Instruction and advice in questions of Rheology and Elasticity by Dr. H. Hencky and Professor W. Hovgaard are gratefully acknowledged.

The research described in this report was performed at the Massachusetts Institute of Technology. For co-operation the author is indebted to Professor Glennon Gilboy and Messrs. E. F. Bennett, B. K. Hough, Jr., R. R. Philippe, and D. W. Taylor, of the research staff.

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# DISCUSSION OF DR. JÜRGENSON'S PAPERS, ENTITLED "THE APPLICATION OF THE THEORY OF ELASTICITY AND THEORY OF PLASTICITY TO FOUNDATION PROBLEMS," AND "RESEARCH ON THE SHEARING RESISTANCE OF SOILS"

By Dr. Arthur Casagrande, Member*

Dr. Jürgenson has presented in his papers an outstanding contribution to our understanding of the behavior of clays under shearing stresses, and has shown us new ways to apply certain little known solutions on the stress distribution in elastic and plastic bodies to problems in foundation engineering. A thorough study of these papers is very stimulating and heartily recommended to all interested in this subject, either for the purpose of application or from the standpoint of research.

Of particular interest to me are the solutions given for the effect of a rigid base, underlying a soft stratum, on the stress distribution in the soft stratum. This is a case encountered often in computations of settlements of structures resting on a deposit of clay. One would be inclined to assume that the presence of a relatively incompressible stratum, *e.g.*, bedrock, will cause an appreciable concentration of the normal stresses on the boundary surface between the hard and the soft stratum. However, according to Dr. Jürgenson's computations, this is not the case. Assuming that sufficient shearing forces can be transmitted along the boundary, so as to prevent any horizontal sliding of the soft material over the hard layer, the presence of this substratum actually causes a spreading of the normal stresses over a wider area and a corresponding decrease in the largest normal stress on the boundary. When a frictionless boundary is assumed, a case which is not likely to be found in nature, the anticipated stress concentration does take place, but only to a slight degree. Evidently there are two different effects produced by the presence of an incompressible base which partially compensate each other. The obvious effect is the reduction of the

* Assistant Professor of Civil Engineering, Graduate School of Engineering, Harvard University.

deflection along the boundary layer due to the smaller compressibility of the harder layer. This reduction is accomplished by the application of upward forces which tend to push the soft layer back into its original position. The reaction is produced by the hard stratum, and results in a higher stress concentration on the boundary below the loaded surface. The other effect is caused by the frictional resistance along the boundary surface which tends to prevent the soft layer beneath the loaded area from deforming laterally, with the result that the load is spread over a larger area. This restraining action can be compared to that of steel reinforcement in the tension zone of a concrete beam, although the analogy is not perfect.

If we assume a case where this boundary is replaced by a surface in which each point can move vertically and not horizontally, and which is underlain by the same soft material that lies above it, then the first effect disappears entirely and the second effect, the reduction in maximum normal stress on this layer, should be expected to become more pronounced. It would be very important to know more about the effects of such layers on the stress distribution, since conditions of this type occur very frequently in nature; for example, the presence of one or more thin sand layers between thick layers of homogeneous clay; or varved clays containing innumerable sand and silt partings, each one representing a surface with greater shearing resistance than that of the clay. If a mathematical solution of these problems cannot readily be found, it would be very desirable to study them with experimental methods, such as photo-elasticity, using thin metal plates fastened to celluloid plates to provide the layers with high shearing resistance. Another possible approach would be by load tests on a medium consisting of alternating layers of gelatine and sand, comparing these with load tests on a thick layer of gelatine alone.

A similar problem is the stress distribution in a thick bed of clay which is overlain by a layer of sand with a thickness of the same order of magnitude as the width of the loaded area. There is no doubt that the presence of a much harder layer over a softer layer results in a wider stress distribution and a corresponding decrease of maximum stresses in the softer layer. It is possible that a settlement analysis, based on the usual assumptions of stress distribution through a homogeneous elastic medium, results in values for the predicted settlements which, in extreme cases, are several times those that will take place in nature due to the smaller loads actually imposed upon the clay. This problem also could be investigated experimentally.

The comparison which Dr. Jürgenson drew between the Swedish method for analyzing deep slides that cut below the toe of an embankment and the shearing stresses computed by the theory of elasticity is very instructive. Engineers should be encouraged to carry out such computations whenever soft soil is loaded with earth structures (embankments, dikes, dams). In this connection I should like to make one suggestion which may be helpful for determining the worst stress condition in the loaded ground beneath certain types of fills. Taking as an example the embankment of approximately triangular cross section, we know from theoretical considerations that the settlement due to con-

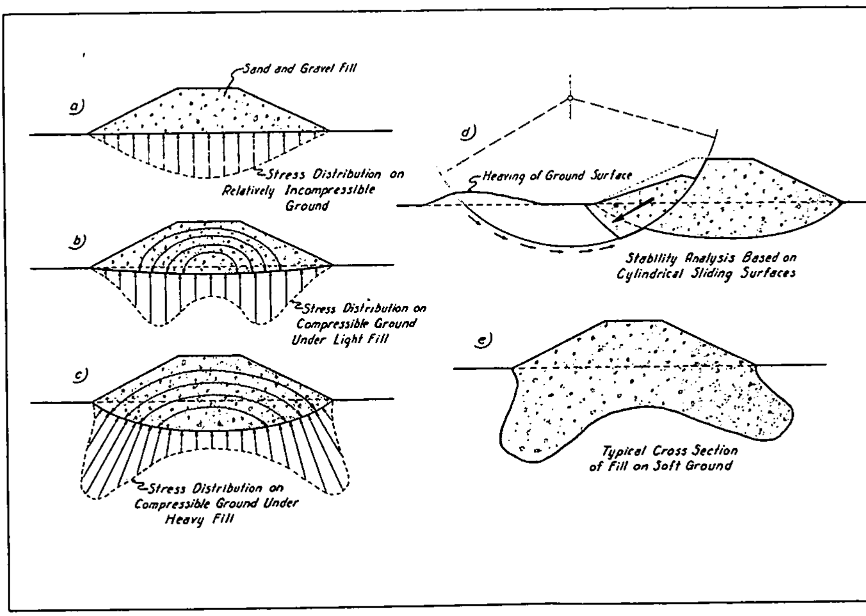


FIG. 1

solidation and deformation of the underground should be largest in the center and smallest along the toes of the embankment as long as the shearing strength of the ground is not approached too closely. In the usual granular types of soil used for such fills this non-uniform settlement will result in an arching action, as shown in Fig. 1b, which will produce stress concentrations near the toes of the fill and relieve the pressures along its center portion. The stresses produced by such arching action on the boundary between fill and subsoil will be almost vertical if the underground is very soft and the fill has not pressed itself into the ground to any appreciable depth. However, when a condition, as shown in Fig. 1c, is reached, the reactions will be concentrated even nearer the edges of the fill and will be inclined, thus tending to push the soil not only indirectly but also directly to the sides. It is, therefore, concluded that the intensity of arching action will depend largely on the character of the soil beneath the fill. As long as the stresses in this soil do not approach its shearing strength, the load of the fill will distribute itself in such a manner that, at any time, the stress distribution in the fill as caused by the settlement, and the settlements produced by the load distribution on the subsoil, will be in equilibrium. During the time these settlements proceed, the stress distribution in the fill itself, and, consequently, also in the underlying soil, changes with time, tending eventually to approach a stationary condition.

If the stresses produced in the underlying soil reach its shearing resistance, a portion of this soil may fail either by displacement along a shearing surface or by plastic flow. This rapid yielding of the underground will destroy the arching action in the fill and suddenly change the stress distribution on the boundary between fill and subsoil. The load will temporarily assume a more normal distribution, that is, largest at the center of the fill. In other words, during the process of yield or failure of a portion of the subsoil those forces which are the cause of it decrease simultaneously, and therefore the deformation or rupture of the soil should stop in its initial stage. In fact, during the construction of such fills on soft ground one does observe very frequently the formation of small cracks along which the outer section of the fill has settled relative to the inner section by an amount varying between a fraction of an inch and several inches. When such a condition is reached, the stresses in the subsoil fluctuate between wide limits as a result of the frequent formation and breakdown of arching within the fill. The occurrence of such minor failures in the underground must be considered an indication of the possibility of serious failures. This is especially true for fills which have settled a considerable distance into the underlying soil, as shown in Fig. 1c, and where that soil has the characteristic of softening upon remolding. The dish-shaped boundary between fill and subsoil permits the building up of a much more effective arching action, and of a thrust against the soil which is sufficiently inclined even to double the magnitude of the lateral force, if compared with that due to a purely vertical reaction. The fact that the inertia of a moving mass increases in direct proportion to the cross-sectional area of a slide, while the resisting forces increase only with the square root of the area, combined with the drop in shearing resistance along the shearing surface as soon as rupture has taken place, are the principal reasons why such larger slides have a tendency to continue after they have started, in spite of the drop in the magnitude of the driving forces associated with the breakdown of the arching action in the fill. Fig. 1d shows such a large slide in progress.

The exploration of fills on soft ground after construction often indicates a condition as shown in Fig. 1e. This may be due either to large slides or to gradual subsidence of the fill. In the latter case the effect of arching action is also very apparent. With increasing subsidence the horizontal soil reaction becomes more effective and permits arching action of sufficient magnitude to bring about the smaller settlement of the center portion.

Valuable observations on the subsidence of fills on soft ground, supporting the above considerations, were made by the Swedish Geotechnical Commission,* Thord Brenner,† and the United States Bureau of Public Roads.‡

The case shown in Fig. 1d was analyzed by means of the Swedish method of cylindrical sliding surfaces and the result compared with the maximum shearing resistance according to the theory of elasticity, using the cross section of the fill as the load diagram. According to the first method, the shearing stress, uniformly distributed along the shearing surface, is about twice that computed by the second method. It is most likely that a stress analysis in the soil, for the case shown in Fig. 1d, on the basis of the theory of elasticity, would yield even larger shearing stresses. This example shows sufficiently the importance of the question of actual load distribution on the boundary between fill and subsoil.

A theoretical approach to this problem has very little chance of success, because so far we have neither a satisfactory solution for the stress distribution in slopes for materials following Hooke's law, nor for the stress distribution beneath a loaded area in granular materials. Well-conducted experimental investigations of this question (*e.g.*, sand fills on gelatine) may yield valuable information. Such information would permit estimates for the most unfavorable stress concentrations that may build up near the edges of fills consisting of cohesionless soils. In the absence of any such information one is obliged to estimate the worst possible stress concentrations purely on the basis of consideration of possible stress distributions in the fill, using Mohr's circle of stress as a criterion. The shearing stresses in the subsoil are then to be determined

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* Statens Järnvägars Geotekniska Kommission: Slutbetänkande. Stockholm, 1922.

† Brenner, Thord: Beispiele von Massenverdraengung durch Bodenbelastung. Fennia 50, No. 19, Helsingfors, 1928.

‡ Aaron, Henry: A Study of Hydraulic Fill Settlement. Public Roads, Vol. 15, No. 1, March, 1934.

either with the help of the theories of elasticity and plasticity, as outlined in Dr. Jürgenson's paper, or by means of cylindrical sliding surfaces.

In his paper on the shearing resistance of clays, Dr. Jürgenson has presented a wealth of new material and ideas. Of great significance is his development of a new type of shearing test, based on Professor Hencky's theoretical investigations of plastic flow. In my opinion this new testing method, which was named by the author "squeezing test," will become an important aid, not only for the research worker but also for routine testing of soils for engineering purposes. It may become especially useful for soil classification, either in conjunction with the liquid limit test, or by replacing the liquid limit test entirely.

To the casual reader it may seem as if the interpretation of the results of direct and indirect shearing tests is becoming increasingly complicated with the increasing refinement in the technique of soil testing. To dispel such discouraging views the following remarks are perhaps not out of place.

According to Terzaghi's theory of consolidation, which forms the basis of the mechanics of fine-grained soils, a mass of soil is defined as consolidated when only hydrostatic pressure is acting in the water that is contained in the voids. If additional stresses are applied, they are at first carried by the water and then gradually transferred upon the solid during a process of volume decrease in which part of the water is pressed out. Let us assume that a clay sample contained in a shearing frame and consolidated under known arbitrary vertical stress is subjected to a shearing stress. This causes an increase in the principle stresses and a change in their direction, to which the sample is not adjusted, and, as a result, part of the stresses are temporarily transferred into the water. Since the water pressure acts at any point equally in all directions (although differing from point to point and varying at any given point with time during the consolidation process), a part of the vertical stress will be carried temporarily by the water. The corresponding reduction in vertical stress from grain to grain will also reduce the effective shearing resistance. From these considerations it appears that the shearing resistance of a clay cannot be constant, but should depend on the rate with which the shearing force is applied. Investigations carried out under my supervision in 1931–32* have demonstrated the great importance of the temporary transfer of stress from the solid phase to the liquid phase during the shearing process.

* A. Casagrande and S. G. Albert: Research on the Shearing Resistance of Soils. Report, Massachusetts Institute of Technology, September, 1932.

In addition to these consolidation effects there seems to be another factor influencing the shearing resistance of clays which Dr. Jürgenson observed from shearing tests on clay samples which were repeatedly remolded. From these tests he concluded that if, after a deformation, the original stress conditions are restored, clay will undergo additional consolidation. How far a slight deformation of an undisturbed mass of clay will result in additional consolidation remains to be further investigated. We do know from a number of observations that the quantity of "lost ground" resulting from excavations is not always sufficient to account for the magnitude of subsidence of adjacent areas and its progress with time. The importance of this question cannot be over empha-

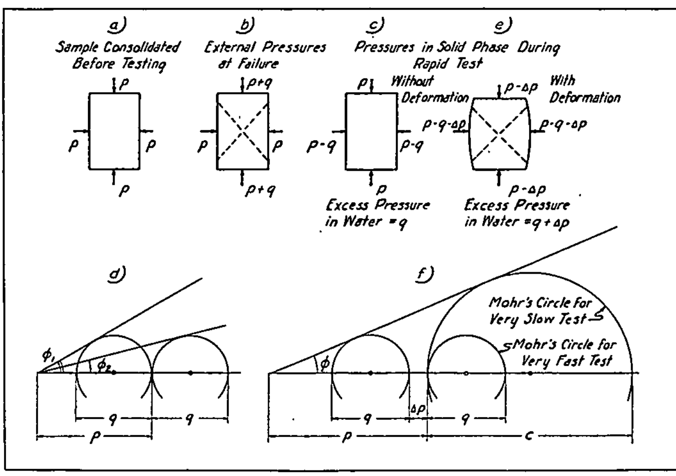


FIG. 2

sized. In the following paragraphs a suggestion is offered for a relatively simple experimental investigation of this question.*

In Fig. 2a is shown a cylindrical sample of clay which we assume to be consolidated under a pressure $p$ acting equally in all directions. Then the vertical pressure is raised quickly until failure occurs at a vertical pressure $p+q$, as indicated in Fig. 2b. Since no time is allowed for consolidation, the additional pressure $q$ will be carried by the water, which means that the horizontal pressure $p$ will no longer be transferred from grain to grain, but only the difference $(p-q)$, $q$ being carried by

* A similar method was used in 1932 to compute the minimum relative angle for a rapid horizontal shearing test. Assuming a true angle of internal friction, I arrived at the smallest possible relative angle of 16 degrees. However, depending on certain conditions, neglected in these computations, this angle may be appreciably smaller or larger.

the water. In other words, the application of an additional vertical pressure has theoretically reduced the horizontal pressure between the soil grains by an equal amount, as shown in Fig. 2c. If we wish to plot Mohr's circle of stress for the condition at failure, we must distinguish between case I, the actual stresses in the solid phase, and case II, the total stress carried by both the solid and the liquid phase. The first case, shown in Fig. 2d by the left circle, yields the true angle of internal friction, $\phi_1$, and the second case, represented by the right circle, gives the relative angle of internal friction, $\phi_2$, for a rapid shearing test. Assuming $\phi_1 = 30°$, a simple computation yields the corresponding $\phi_2 = 14° \ 30'$.

In these considerations we have neglected the fact that, due to the deformation, the sample is no longer consolidated under a pressure $p$, but under a smaller pressure $(p - \Delta p)$. The problem resolves itself into the determination of $\Delta p$. Fig. 2e shows the external and internal stress conditions of the sample at failure, and in Fig. 2f the stresses are presented by means of Mohr's diagram. If on an identical sample a similar test is conducted at a sufficiently slow rate to allow consolidation for each new pressure increment, the compressive strength $c$ will be much larger than the compressive strength $q$ in the fast test, and, since no hydrodynamic stresses are acting, the corresponding angle $\phi$ (see Fig. 2f) will be the true angle of internal friction. The value of $\Delta p$ can then be determined graphically, as shown in Fig. 2f.

In conclusion I should like to express the hope that Dr. Jürgenson will continue work along these lines and furnish us with answers to many of the questions which present themselves when reading his stimulating papers.

# Of General Interest

## NEW RELATIONSHIP BETWEEN BOSTON SOCIETY OF CIVIL ENGINEERS AND THE ENGINEERING SOCIETIES OF NEW ENGLAND

A change of considerable importance in the relationship between the Boston Society of Civil Engineers and the Engineering Societies of New England has resulted from the alteration of the aims of the Engineering Societies, namely, to undertake only such activities and to occupy only such office space as may be necessary for carrying out those functions which may benefit the whole affiliation of Engineering Societies, and to require each separate organization to provide its own quarters and clerical staff.

The Boston Society of Civil Engineers, after careful consideration by the Board of Government, will continue to occupy quarters at 715 Tremont Temple, somewhat reduced in area; but in keeping with its aims this will provide office accommodations, space for some meetings and the library.

By an agreement with the Engineering Societies of New England, that organization will maintain an office in a portion of the area retained by the Boston Society of Civil Engineers at an agreed-upon rental, and this will afford, also, the privilege of the reading rooms and library for all members of the Engineering Societies.

A reduction in pro rata dues paid by Boston Society of Civil Engineers to the Engineering Societies will offset the cost for rental of office space and clerical services, so that there will be no additional expense to be borne by the Boston Society of Civil Engineers.

With the decision of the New England Water Works Association, many of whose members are also members of our Society, to rent other quarters than Tremont Temple, and the future duties of the Engineering Societies of New England now limited to those which they can best perform for the individual societies, the Board of Government now feels that many of the objections to membership in the Engineering Societies of New England and the present quarters will become less pertinent. The present arrangements need not be final, but they offer a satisfactory solution to a somewhat unexpected situation.

## Joint Outing, New England Water Works Association and Boston Society of Civil Engineers

The regular meeting of the Boston Society of Civil Engineers was omitted in June in order to join with the New England Water Works Association in an outing which was held at the Salem Country Club, Peabody, on Wednesday, June 27, 1934.

A program of sports, games of bridge and golf was enjoyed by many during the afternoon. The dinner, held at 6.30 P.M., was attended by about 200 members, ladies and guests. Prizes were distributed after dinner, and dancing concluded the program.

The committees in charge were A. A. Ross, Chairman, R. W. Estey, Assistant Chairman, for the New England Water Works Association, and Everett N. Hutchins and J. H. Harding for the Boston Society of Civil Engineers.

# PROCEEDINGS OF THE SOCIETY

## MINUTES OF MEETINGS

### Boston Society of Civil Engineers

APRIL 16, 1934. — A regular meeting of the Boston Society of Civil Engineers was held this evening at the Engineers' Club, and was called to order by the President, Arthur T. Safford, at 7 P.M. Forty members and guests attended the meeting, preceded by a buffet supper with thirty attending.

The President introduced the speaker, Hon. Henry F. Long, Commissioner of Taxation and Corporations, who gave a very interesting talk on the general subject of taxation. An informal question period followed the talk.

Adjourned at 9.20 P.M.

EVERETT N. HUTCHINS, *Secretary.*

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MAY 16, 1934. — A regular meeting of the Boston Society of Civil Engineers was held this evening at the Engineers' Club, and was called to order by the President, Arthur T. Safford, at 7 P.M. Seventy-one members attended the meeting, preceded by a buffet supper with fifty-nine attending.

The President announced the death of the following members: James H. Sullivan, died on April 4, 1934, a Member since November 16, 1910; Arthur D. Marble, died on April 30, 1934, a Member since December 19, 1883; Herbert A. Wilson, died on May 7, 1934, a Member since May 17, 1897.

The President stated that the joint outing with the New England Water Works Association will be held at the Salem Golf Club on Wednesday, June 27, 1934, and that the regular June meeting of the Society would be omitted.

The President introduced the speaker, Dr. Glennon Gilboy, Associate Professor of Soil Mechanics, Massachusetts Institute of Technology, who gave a talk on "Mechanics of Hydraulic Fill Dams," illustrated by lantern slides.

An informal question period followed the talk.

Adjourned at 9.10 P.M.

EVERETT N. HUTCHINS, *Secretary.*

### Designers Section

APRIL 11, 1934. — The Designers Section of the Boston Society of Civil Engineers met in the Society rooms in Tremont Temple on Wednesday evening at six o'clock, April 11, 1934.

The minutes of the annual meeting, held on March 14, were read and approved.

Mr. A. B. MacMillan of the Aberthaw Construction Company presented an interesting paper on "Construction Features of the New Christian Science Publishing House," a momumental building. He pointed out the savings in the method of handling the ground water by the well point system. The placing of the monolithic, 15-ton columns, 21 feet long, in the portico was an interesting feature.

An enthusiastic discussion followed the presentation of the paper.

There were present nineteen members and guests. The meeting adjourned at 7.40 P.M.

ALBERT HAERTLEIN, *Clerk.*

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MAY 9, 1934. — The Designers Section of the Boston Society of Civil Engineers met in the Society rooms in Tremont Temple on Wednesday evening, May 9, 1934.

The minutes of the meeting held on April 11, 1934, were read and approved.

Dr. Leo Jürgenson, Research Associate in Soil Mechanics at Massachusetts Institute of Technology, presented a paper entitled "The Application of the Theory of Elasticity and the Theory of Plasticity to Foundation Problems."

There were present twenty-one members and guests. The meeting adjourned at 7.45 P.M.

ALBERT HAERTLEIN, *Clerk.*

## Highway Section

FEBRUARY 28, 1934. — The annual meeting of the Highway Section of the Boston Society of Civil Engineers was held at the rooms of the Engineering Societies, 715 Tremont Temple, with Chairman Kleinert presiding.

The nominating committee reported and presented the following nominations for officers of the Section and members of the Executive Committee for the year 1934–35:

Chairman — Albert Haertlein.
Vice-Chairman — Arthur P. Rice.
Clerk — Alexander J. Bone.
Additional Members of Executive Committee — H. F. Heald, C. F. Knowlton and L. S. Wertheim.

The report of the Nominating Committee was accepted and the Chairman instructed to cast one ballot for the election of all officers as submitted.

The Committee on Relation of Sections to Main Society presented a draft of By-Laws of the Sections, which were unanimously adopted by the Section.

The paper of the evening was "Research on Frost Action in Soils and its Effects on Highways," by Dr. Arthur Casagrande of Harvard University. Dr. Casagrande described investigations which he has made both in the field and laboratory to determine the effect of frost on pavements of different types and sub-soil conditions.

There were twenty-five members of the Section and their guests present. The meeting adjourned at 9.15 P.M.

ALEXANDER J. BONE, *Clerk.*

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MAY 23, 1934. — A regular meeting of the Highway Section of the Boston Society of Civil Engineers was held at the Society rooms, 715 Tremont Temple, at 7.15 P.M. Chairman Haertlein proceeded at once to introduce Mr. L. E. Andrews, Engineer with the Portland Cement Association, who gave an illustrated talk on "Recent Developments in the Design and Construction of Concrete Pavements." Mr. Andrews spoke particularly of recent trends in joint design, the use of vibrators in concrete road construction, and methods used in the construction of the cement-bound macadam type of pavement.

There were twenty members present. The meeting adjourned at 9.30 P.M.

ALEXANDER J. BONE, *Clerk.*

## Northeastern University Section

MARCH 14, 1934. — The Northeastern University Section of the Boston Society of Civil Engineers held its regular meeting in 18–H of the Huntington Building of Northeastern University on March 14, 1934, at 8 P.M.

Vice-Chairman Pittendreigh presided. He mentioned the forthcoming annual meeting, which will take place in the next Division–B period, and that a nominating committee should be selected to nominate officers.

*Voted,* That the Vice-Chairman appoint a nominating committee from the junior class who are members of the Boston Society of Civil Engineers.

Mr. Pittendreigh then introduced the speaker of the evening, Mr. Albert Genaske, assistant engineer of the Metropolitan District Water Supply Commission. His subject was "The Water Supply Project at Enfield, Mass.," illustrated by moving pictures.

The pictures showed the stream-control tunnel diverting the water around the construction, the caisson construction and operation within the working chamber, the means of getting the water to Boston from the Wachusett Reservoir, and the villages that will be flooded by the waters impounded by the new Quabbin Reservoir. His very interesting talk was well received by the thirty-five members and guests present.

The meeting adjourned at 9.45 P.M.

FREDERICK HALL, *Clerk.*

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APRIL 26, 1934. — The annual meeting of the Northeastern Section of the Boston Society of Civil Engineers was held on April 26, 1934, in Room 18–H, Huntington Hall, and was called to order at 8 o'clock P.M.

The following officers were elected for the coming year:

Chairman — James N. DeSerio.
Clerk — Kenneth F. Knowlton.
Executive Committee — Richard L. Dunning, John H. Wakenigg and Robert R. Balmer, Jr.

A very interesting talk was given by Mr. Harry E. Sawtell, Associate, Charles T. Main, Inc., on the soil conditions underneath the buildings at Massachusetts Institute of Technology, illustrated by lantern slides.

There were twenty members present.

BRUCE SCHOW, *Clerk.*

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## ADDITIONS

*Students*

RALPH F. D'ELIA, 41 Marion Street, Medford, Mass.

G. ALBION SMITH, 8 Swan Street, Beverly, Mass.

ROGER B. STEVENSON, Cummaquid, Mass.

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## DEATHS

| | |
|---|---|
| GEORGE W. FULLER | June 15, 1934 |
| FRANK F. JONSBERG | June 14, 1934 |
| ARTHUR D. MARBLE | April 30, 1934 |
| JAMES H. SULLIVAN | April 4, 1934 |
| HERBERT A. WILSON | May 7, 1934 |

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BOSTON SOCIETY OF CIVIL ENGINEERS

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Charles B. Breed
Frank E. Winsor
Lewis E. Moore

A. B. Edwards
Frank L. Flood
Lawrence G. Ropes
(Term expires March, 1935)

Edward S. Averell
Kenneth C. Reynolds
Frederic N. Weaver
(Term expires March, 1936)

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J. Stuart Crandall
George H. Delano
Carroll A. Farwell
Albert Haertlein
Ralph W. Horne
Everett N. Hutchins
C. Frederick Joy, Jr.
S. Stanley Kent
Charles R. Main
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John B. Babcock
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Edward S. Averell
Kimball R. Garland
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