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## SOME OBSERVATIONS ON DIMENSIONAL ANALYSIS

(Presented at a meeting of the Hydraulics Section of the Boston Society of Civil Engineers held on May 6, 1942.)

By FRANKLIN O. ROSE*

ONE of the principal benefits to be derived from the use of dimensional analysis is in the power which it gives us to set up a functional relationship between the physical quantities which control some physical phenomenon and to note what measurements must be made of these physical quantities in order that the exact form of the function may be determined. There are two ways of setting up this functional relationship—the use of the Rayleigh method and the use of Buckingham's $\pi$-theorem. In order to refresh the memory somewhat on this subject, I shall work out a very simple problem by each method, and then I should like to discuss the $\pi$-theorem in particular and try to answer some of the questions which arise in the minds of those who use it.

The illustrative problem is taken from some experimental work which has been carried on at the Massachusetts Institute of Technology, in connection with the effect of angularity of approach upon the discharge coefficient for a dam with a roller gate. For simplicity, the problem will be confined to a consideration of the coefficient in the case of but one angle of approach.

It was assumed that $q$, the discharge per unit length of the gate, is a function of $d$, the height of the gate opening, $h$, the net head be-

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tween the upper and lower pools, plus velocity-head of approach, and $g$, the acceleration of gravity. By the Rayleigh method, this assumption would be expressed

$$q = f(d, h, g) \text{ or } q = K\ d^x h^y g^z$$

and the exponents, $x$, $y$, and $z$, would be determined by equating the exponents of similar fundamental units on either side of the equation.

The four physical quantities appearing in the functional relationship can be expressed in terms of the fundamental units of length and time, as follows:

$$\begin{aligned} q &= L^2/T \\ d &= L \\ h &= L \\ g &= L/T^2 \end{aligned}$$

Then

$$L^2/T = L^x\ L^y\ (L/T^2)^z$$

and

$$\begin{aligned} x + y + z &= 2 \\ -2z &= -1 \end{aligned}$$

from which

$$\begin{aligned} z &= 1/2 \\ x &= 3/2 - y \end{aligned}$$

and

$$q = K\ d^{3/2\ -\ y}\ h^y\ g^{1/2}$$

or

$$q = K\ d^{3/2} \sqrt{g}\ (h/d)^y$$

or

$$q/d^{3/2} \sqrt{g} = K\ (h/d)^y$$

with dimensionless quantities on each side of the equation.

In handling this problem by means of the $\pi$-theorem, the previous assumption would be expressed:

$$F\ (d, q, h, g) = 0$$

$$\pi_1 \text{ would be written as } d^a g^b q = L^a \left(\frac{L}{T^2}\right)^b \frac{L^2}{T}$$

and, as this $\pi$-term is to be dimensionless,

$$L^a \left(\frac{L}{T^2}\right)^b \frac{L^2}{T} = L^o T^o$$

and

$$a + b + 2 = 0$$
$$-2b - 1 = 0$$

from which

$$b = -\tfrac{1}{2} \text{ and } a = -3/2$$

and

$$\pi_1 = \frac{q}{d^{3/2} g^{1/2}}$$

Similarly,

$$\pi_2 = d^u g^v h = L^u \left(\frac{L}{T^2}\right)^v L = L^o T^o$$

and

$$u + v + 1 = 0$$
$$-2v = 0$$

from which

$$v = 0 \text{ and } u = -1$$

and

$$\pi_2 = \frac{h}{d}$$

and the $\pi$-theorem states that $\phi(\pi_1, \pi_2) = 0$ or $\phi\left(\frac{q}{d^{3/2} g^{1/2}}, \quad \frac{h}{d}\right) = 0$

or

$$\frac{q}{d^{3/2} g^{1/2}} = \psi \frac{h}{d}$$

This result is seen to be identical with that obtained by the Rayleigh method. The reasoning behind the Rayleigh method is, I believe, quite obvious, but it is not so with the $\pi$-theorem and the latter is what I should like to discuss a little more in detail. First, however, I shall complete the illustrative problem.

The results of the dimensional analysis showed what physical measurements would have to be made in the laboratory. After these measurements had been made and the dimensionless quantities $\frac{q}{d^{3/2} g^{1/2}}$ and $\frac{h}{d}$ computed for four runs, with only $q$ and $h$ varied, these quantities, when plotted against each other logarithmically, yielded a straight line, the equation of which was

$$\frac{q}{d^{3/2}\, g^{1/2}} = 1.07\left(\frac{h}{d}\right)^{1/2}$$

$$\text{or } q = .76\, d\, \sqrt{2g\, h}$$

$$\text{or } Q = .76\, A\, \sqrt{2g\, h}$$

for the particular angularity of approach in the part of the experiment used for this illustration. $Q$ is the total discharge and $A$ is the total area of the gate opening.

E. Buckingham, in the 1915 Transactions of the American Society of Mechanical Engineers, gave his deduction of the $\pi$-theorem, which is probably more clear to the mathematician than to the average engineer; in fact, I would hazard the guess that perhaps nine out of ten engineers who use it more or less accept its validity. Certainly, I have seen nothing but its bare acceptance in any text book in which it is mentioned.

One of my objects is to attempt to make quite clear Buckingham's reasoning. I shall do this by quoting freely from, or paraphrasing, his deduction of the theorem, and paralleling each step with a similar step for a specific example.

Any physical phenomenon, Buckingham says, can be completely represented by an equation, analytically or empirically determined, of the form

$$F\ (Q_1, Q_2, \ldots\ldots\ldots Q_n) = 0 \qquad (1)$$

in which each of the various $Q$'s, $n$ in number, represents a physical measurable quantity, such as a velocity, a density, a dimension of length, etc. By 'completely' is meant that the value of any one of the $Q$'s is completely determined when the values of the others are known. As was shown in the example of the dam with a roller gate, the form of equation (1) can be determined by use of the $\pi$-theorem.

The particular equation which I have chosen with which to parallel the general reasoning is the familiar Poiseuille formula for lost head in a pipe when the flow is laminar. This equation is

$$h_F = \frac{32\mu l v}{w d^2} = \frac{32\mu l v}{\rho g d^2}$$

or

$$h_F - \frac{32\mu l v}{\rho g d^2} = 0$$

or

$$F\ (d, v, \mu, \rho, l, h_f, g) = 0 \qquad (1')$$

which is recognized as of the form of equation (1).

In this equation, $d$ is the diameter of the pipe, $v$, $\mu$, and $\rho$ are the velocity, viscosity and density, respectively, of the fluid, $l$ is the length of pipe in which the lost head, $h_f$, occurs and $g$ is the acceleration of gravity.

Any physical equation, such as (1), has the general form

$$\Sigma\, M\, Q_1{}^a\, Q_2{}^b \,\ldots\ldots\ldots\, Q_n{}^n = 0 \qquad (2)$$

which represents a summation of terms, each term being a product of a dimensionless coefficient, $M$, and the $Q$'s of equation (1) with certain exponents, which may be anything, including zero.

$$h_F - \frac{32\mu l v}{\rho g d^2} = 0 \quad \text{may be written}$$

$$d^0 v^0 \mu^0 \rho^0 l^0 h_f\, g^0 + (-32)\, d^{-2} v \mu \rho^{-1} l\, h_f{}^0 g^{-1} = 0 \qquad (2')$$

which is seen to be our particular case of equation (2), the summation being of two terms.

The dimensions of each term in equations (2) and (2′) must be the same, as it is not possible to add or subtract terms of unlike dimensions. For instance, a velocity cannot be added to a density. In (2′) the dimension of each term is that of a length and the numerical coefficients, 1 and —32, are dimensionless. The exponents $a, b, \ldots\ldots\ldots n$ of equation (2) have become 0, 0, 0, 0, 0, 1, 0 and —2, 1, 1, —1, 1, 0, —1 for the two terms of equation (2′).

If the general equation (2) were divided through by any one of its terms, the result would be

$$1 + \Sigma\, N\, Q_1{}^\alpha\, Q_2{}^\beta \,\ldots\ldots\ldots\, Q^\nu{}_n = 0 \qquad (3)$$

in which each term must be dimensionless, since each term is a ratio of the terms of similar dimensions of equation (2), and the $N$'s are obtained by dividing one dimensionless $M$ by another.

Specifically, upon dividing equation (2′) through by its first term, there results

$$1 + (-32)\ d^{-2} v \mu \rho^{-1} l\ h_f{}^{-1} g^{-1} = 0 \qquad (3')$$

and here the summation is of but one term.

We will now leave equations (3) and (3′) for a moment, and return to them later.

To measure $n$ kinds of quantity, we require $n$ units, but these need not all be adopted arbitrarily for they can in general be derived from (described or defined in terms of) some smaller number of fundamental units which are independent of each other. In mechanics, of which hydraulics is a branch, all of the necessary units can be defined in terms of three, which would be considered fundamental, such as mass, length and time; force, length and time; work, velocity and density, etc. For instance, if the fundamental units adopted were mass, length and time, force could be expressed dimensionally as $ML/T^2$ and, if the fundamental units were work, velocity and density, force could be expressed as $\sqrt[3]{W^2v^2\rho}$. The fundamental units of mass, length and time will be used throughout this discussion. In phenomena involving heat or electricity, additional fundamental units are necessary.

In the specific example chosen to illustrate the $\pi$-theorem, the seven (or $n$) kinds of quantity are $d$, $v$, $\mu$, $\rho$, $l$, $h_f$, and $g$ and these seven may all be expressed dimensionally in terms of the fundamental units adopted, as follows: $d = L$, $v = L/T$, $\mu = M/LT$, $\rho = M/L^3$, $l = L$, $h = L$ and $g = L/T^2$.

If, in the general case, we let $k$ be the number of fundamental independent units required for defining the $n$ kinds of unit needed in measuring the quantities $Q_1, Q_2, \ldots\ldots\ldots Q_n$, we can express a set of $k$ of these units as $Q_1, Q_2, \ldots\ldots\ldots Q_k$ and the remaining $n$-$k$ units could be denoted by $P_1, P_2, \ldots\ldots\ldots P_{n\text{-}k}$. Paralleling this with the illustrative example, the three fundamental units might be taken as $d$, $v$, $\mu$, with the four remaining ones being $\rho$, $l$, $h_f$, $g$.

Then each of the $P$'s may be derived from the $Q$'s in accordance with a dimensional equation which may be written

$$Q_1^{\alpha}\, Q_2^{\beta} \ldots\ldots\ldots Q_k^{\kappa}\; P = M^0L^0T^0 = [1] \tag{4}$$

or

$$d^x v^y \mu^z \rho = M^0L^0T^0 = [1] \tag{4'}$$

The foregoing means simply this: if it were desired to express $\rho$ dimensionally in terms of $v$, $d$ and $\mu$, there could be written the equation

$$\rho = d^a v^b \mu^c \qquad \text{or } \frac{M}{L^3} = L^a\left(\frac{L}{T}\right)^b\left(\frac{M}{LT}\right)^c$$

where the exponents, $a$, $b$ and $c$, must be such that exponents of like units on either side of the equation would be the same. With this being true and upon dividing the equation through by its right-hand member, there would result

$$d^x v^y \mu^z \rho = [1]$$

which is to say that, with the proper values for $x$, $y$ and $z$, the quantity $d^x v^y \mu^z \rho$ is dimensionless.

Since there are $n$-$k$ of the $P$'s, there are $n$-$k$ separate independent equations of the form of (4), and no more. Now if, in each of these $n$-$k$ equations, we substitute for $P$ and the $Q$'s their dimensional equivalents in terms of any fundamental units, the requirement that the total exponent of each fundamental unit shall vanish, or equal zero, furnishes $k$ independent linear equations which will suffice for the determination of the exponents $\alpha, \beta \ldots\ldots\ldots \kappa$.

To continue the illustrative example, equation (4′) may be written

$$L^x \left(\frac{L}{T}\right)^y \left(\frac{M}{LT}\right)^z \frac{M}{L^3} = M^0 L^0 T^0$$

and, equating exponents of the fundamental units to zero, it is seen that we have three independent linear equations for the determination of $x$, $y$, and $z$.

$$\begin{aligned} &\text{For } M\text{:} \quad z + 1 = 0 \\ &\text{For } L\text{:} \quad x + y - z - 3 = 0 \\ &\text{For } T\text{:} \quad -y - z = 0 \end{aligned}$$

Solving,

$$\begin{aligned} x &= 1 \\ y &= 1 \\ z &= -1 \end{aligned}$$

and equation (4′) becomes

$$\frac{dv\rho}{\mu} = [1] \text{ or the term } \frac{dv\rho}{\mu} \text{ is dimensionless.}$$

If, after determining the exponents for any $P$, we call equation (4) $\pi$, we have

$$\pi = Q_1^{\alpha} Q_2^{\beta} \ldots\ldots\ldots Q_k^{\kappa} P$$

and it is seen that $\pi$ is a dimensionless product of some of the $Q$'s of equation (1). Since there are $n$-$k$ independent equations of the form of (4), there is the same number of independent $\pi$'s, which is $k$ less than the total number, $n$, of different kinds of quantity, $Q$. These dimensionless products may be denoted by the symbols $\pi_1, \pi_2 \ldots\ldots \pi_i$ and may be written in dimensional form.

$$\pi_1 = Q_1^{a_1} Q_2^{\beta_1} \ldots\ldots Q_n^{\nu_1}$$
$$\pi_2 = Q_1^{a_2} Q_2^{\beta_2} \ldots\ldots Q_n^{\nu_2}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\pi_i = Q_1^{a_i} Q_2^{\beta_i} \ldots\ldots Q_n^{\nu_i}$$

and one set of $\pi$'s, four in number $(i = n\text{-}k = 7 - 3 = 4)$ may be written for the illustrative example, as follows:

$$\pi_1 = d^{-1}v^{-1}\mu\rho^{-1}l^0 h_F{}^0 g^0 = \frac{\mu}{dv\rho}$$
$$\pi_2 = d^{-1}v^2\mu^0\rho^0 l^0 h_F{}^0 g^{-1} = \frac{v^2}{dg}$$
$$\pi_3 = d^{-1}v^0\mu^0\rho^0 l\, h_F{}^0 g^0 = \frac{l}{d}$$
$$\pi_4 = d\, v^0\mu^0\rho^0 l^0 h_F{}^{-1} g^0 = \frac{d}{h_F}$$

Other sets of independent $\pi$'s could be obtained by a different choice of the first $k$ quantities in equation (4), or the first three quantities in equation (4′).

Now, these $\pi$'s, being dimensionless products, will remain dimensionless if raised to any power whatsoever and, therefore, going back to equation (3), we see that it could be written in the form:

$$1 + \Sigma N \pi_1^{x_1} \pi_2^{x_2} \ldots\ldots \pi_i^{x_i} = 0 \qquad (5)$$

The illustrative example would yield:

$$1 + (-32)\left(\frac{\mu}{dv\rho}\right)^{x_1}\left(\frac{v^2}{dg}\right)^{x_2}\left(\frac{l}{d}\right)^{x_3}\left(\frac{d}{h_F}\right)^{x_4} = 0 \qquad (5')$$

If the exponents, $x_1$, $x_2$, etc., were all unity, there would result:

$$1 + (-32)\frac{\mu}{dv\rho}\,\frac{v^2}{dg}\,\frac{l}{d}\,\frac{d}{h_F} = 0$$

or

$$1 + (-32)\ d^{-2} v \mu \rho^{-1} l h_F g^{-1} = 0$$

which is equation (3′).

The dimension of any term in equation (5) is unaffected by the number of terms or the numerical value of the coefficient, $N$, and of the exponent, $x$. Therefore, the left-hand side of equation (5) is merely some entirely independent function of the $\pi$'s and may be written

$$\phi\ (\pi_1, \pi_2, \ldots\ldots\ldots \pi_i) = 0 \tag{6}$$

to which corresponds, in the illustrative example,

$$\phi\left(\frac{\mu}{dv\rho}, \frac{v^2}{dg}, \frac{l}{d}, \frac{d}{h_F}\right)' \doteq 0 \tag{6'}$$

In the general development of the theory attention has hitherto been confined to a group of $Q$'s that are all of a different kind. If several quantities of any one kind, such as $d$, $l$ and $h_F$ in the illustrative example, are involved in the relationship to be described, they may all be specified by the value of any one of them and the ratios $r'$, $r''$, etc., between that one and the others. It is seen that each of these ratios is a dimensionless quantity.

Equation (1) could then be written

$$F\ (Q_1, Q_2, \ldots\ldots\ldots Q_{n-r}, r', r'' \ldots\ldots\ldots) = 0 \tag{7}$$

and equation (1′)

$$F\ (d, v, \mu, \rho, \frac{l}{d}, \frac{d}{h_F}, g) = 0 \tag{7'}$$

and equation (6) could be written

$$\phi\ (\pi_1, \pi_2, \ldots\ldots\ldots \pi_{n-k-r}, r', r'' \ldots\ldots\ldots) = 0 \tag{8}$$

with $r$ of the $\pi$'s replaced by the $r$ dimensionless ratios. Thus, in equation (6′) for the illustrative case, instead of four $\pi$-terms, there are two $\pi$-terms and two $r$-terms.

It has now been shown that, if a physical phenomenon can be described by the equation $F\ (Q_1, Q_2, \ldots\ldots\ldots Q_n) = 0$ or $F\ (Q_1, Q_2, \ldots\ldots\ldots Q_{n-r}, r', r'' \ldots\ldots\ldots) = 0$, either the equation $\phi\ (\pi_1, \pi_2, \ldots\ldots\ldots \pi_i) = 0$ or $\phi\ (\pi_1, \pi_2, \ldots\ldots\ldots \pi_{n-k-r}, r', r'' \ldots\ldots\ldots = 0$ is true, and this is the $\pi$-theorem.

Here dimensional analysis ends. Just what this indicated functional relationship among the $\pi$'s is must be determined, either empirically by direct experiment or theoretically from such general physical principles as may be applicable.

One example of the use of dimensional analysis has already been given in the case of the dam with roller-gate. This was a rather simple situation, with only two $\pi$-terms involved. In order to see how the matter is handled when there are more than two $\pi$-terms, let us consider the problem of a ship-model which is towed in a towing tank, with the object of determining the drag. It is assumed that the physical quantities, or the $Q$'s, entering this phenomenon are the drag, $D$, the velocity of the model, the density and viscosity of the fluid, the displacement, $\Delta$, of the model, and the acceleration of gravity. The last quantity enters the problem on account of the gravity waves which are formed and which have so much to do with the resistance offered to the movement of the ship through the water. Expressed mathematically, the foregoing assumption is $F\ (D,\ v,\ \rho,\ \mu,\ \Delta,\ g) = 0$. There are six $Q$'s and $k$ is three; therefore, there will be six minus three, or three, $\pi$'s. Letting

$$\pi_1 = \Delta^x\,\rho^y\,v^z\,\mu = \left(\ L^3\ \right)^x\left(\frac{M}{L^3}\right)^y\left(\frac{L}{T}\right)^z\ \frac{M}{LT}$$

$$y + 1 = 0$$

$$3x - 3y + z - 1 = 0$$

$$-z - 1 = 0$$

from which

$$x = -{}^1/_3,\ y = -1,\ z = -1$$

and $\pi_1 = \dfrac{\mu}{\Delta^{1/3}\rho v} = \dfrac{\mu}{l\rho v}$ which is recognized as the reciprocal of Reynolds' number, $R$.

Letting

$$\pi_2 = \Delta^a\rho^b v^c D = \left(\ L^3\ \right)^a\left(\frac{M}{L^3}\right)^b\left(\frac{L}{T}\right)^c\ \frac{ML}{T^2}$$

$$b + 1 = 0$$

$$3a - 3b + c + 1 = 0$$

$$-c - 2 = 0$$

from which

$$a = -{}^2/_3,\ b = -1,\ c = -2$$

and

$$\pi_2 = \frac{D}{\Delta^{2/3}\rho v^2}$$

Letting

$$\pi_3 = \Delta^d \rho^e v^f g = \left( L^3 \right)^d \left(\frac{M}{L^3}\right)^e \left(\frac{L}{T}\right)^f \frac{L}{T^2}$$

$$e = 0$$
$$3d - 3e + f + 1 = 0$$
$$-f - 2 = 0$$

from which

$$d = {}^1/_3,\ e = 0,\ f = -2$$

and $\pi_3 = \dfrac{\Delta^{1/3}g}{v^2} = \dfrac{lg}{v^2}$ which is also recognized as the reciprocal of Froude's number, $F$.

Then we have

$$\phi\left(\frac{\mu}{l\rho v},\ \frac{D}{\Delta^{2/3}\rho v^2},\ \frac{lg}{v^2}\right) = \phi\left(\frac{1}{R},\ \frac{D}{\Delta^{2/3}\rho v^2},\ \frac{1}{F}\right) = 0 \qquad (9)$$

The expression $y = f\ (\frac{1}{R})$ is the same as $y = k\ (\frac{1}{R})^n$ and the particular function of $\frac{1}{R}$ would be specified when $k$ and $n$ are specified. $n$ might very well have the value of $-1$, in which case we would have $y = kR$ or $y = F\ (R)$. This is to say that equation (9) could be written

$$\theta\left(R,\ \frac{D}{\Delta^{2/3}\rho v^2}\ \ F\right) = 0$$

and that, in general, if a $\pi$-term is evaluated in a certain form, it may be used either in this form or as its reciprocal or, in fact, it could be raised to any power.

Now, returning to the ship-model problem, it should be stated that, in order to have dynamic as well as geometric similarity between model and prototype, it is necessary, in cases where viscous forces and gravity forces are both acting, that both the Reynolds and Froude numbers for model and prototype be, respectively, the same. This requirement is expressed mathematically by

$$\frac{v_p \rho_p l_p}{\mu_p} = \frac{v_m \rho_m l_m}{\mu_m}$$

and

$$\frac{v_p^2}{l_p g_p} = \frac{v_m^2}{l_m g_m}$$

where the subscripts, $p$ and $m$, refer to prototype and model, respectively. As density, viscosity and the acceleration of gravity in the model are equal to the corresponding quantities in the prototype, these equations can be written

$$v_p\, l_p = v_m\, l_m$$

and

$$\frac{v_p^2}{l_p} = \frac{v_m^2}{l_m}$$

from which $\dfrac{v_p}{v_m}$ must equal both $\dfrac{l_m}{l_p}$ and $\sqrt{\dfrac{l_p}{l_m}}$ which is obviously impossible unless $l_m = l_p$ or, in other words, the model be built the same size as the prototype.

This difficulty is overcome by temporarily ignoring the effect of viscosity and conducting the model tests with the Froude numbers equal for model and prototype, and then correcting the results of the tests by data secured from tests where the Reynolds numbers have been made equal.

Another method whereby definite results might be obtained when the number of $\pi$-terms and $r$'s is greater than two is to, if possible, arrange matters in the experimental program so that the $r$'s and all but one or two of the $\pi$'s are kept constant. Equation (8) could be solved for one of the $\pi$'s and written:

$$\pi_1 = Q_1^{\alpha}\; Q_2^{\beta} \ldots\ldots Q_k^{\kappa}\; P_1 = \psi\; (\pi_2, \pi_3 \ldots\ldots \pi_{n\text{-}k\text{-}r}, r', r'' \ldots\ldots\ldots)$$

Then, if the $r$'s and all of the $\pi$'s but $\pi_1$ had been kept constant, we would have $P_1 = N\; Q_1^a\; Q_2^b \ldots\ldots\ldots Q_k^k$ and a single experimental measurement of a simultaneous set of values of $P_1$ and the $Q$'s would determine the coefficient, $N$, which would then enable us to compute the value of $P_1$ for any other values of the $Q$'s which would still keep the $r$'s and the other $\pi$'s constant. Similarly, if all but two of the $\pi$'s had been kept constant, we would have had $P_1 = N\; Q_1^a \cdot Q_2^b \ldots\ldots\ldots Q_k^k\; \psi\; (\pi_2)$, which is similar to the equation finally determined in the example of the dam with roller gate. Here again $N$ and $\psi$ could be determined by logarithmic plotting and $P_1$ could then be computed for any other values of the $Q$'s and $r$'s which would satisfy the requirements of keeping the $r$'s and other $\pi$'s constant.

The possibility, or impossibility, of thus keeping the $r$'s and $\pi$'s constant in any particular problem determines whether or not reliable information can be determined from model tests. It is not always possible.

The question often arises, in this use of dimensional analysis, of how it is known that all of the physical quantities which enter into a given physical phenomenon are included at the beginning of the computation. In such a case, the fact that some quantity had been omitted would be shown by the fact that the resulting dimensionless quantities, when computed from the experimentally determined data and plotted against each other, would not yield a smooth curve. A little reflection will show why this is so.

In the first place, it is quite necessary to realize that dimensional analysis simply makes it possible for us to write equations which are dimensionally correct. There is no physical connection between the pencil which writes such an equation and the phenomenon which that pencil is attempting to describe mathematically which would definitely associate the one with the other. If, however, an equation which does correctly define a physical phenomenon, no matter how that equation had been determined, were plotted, then corresponding measured values of the variables, obtained by experimental means, would fall on that curve. If the equation had been incorrectly determined and then plotted, the measured values would not coincide with the curve. If, through experience and the use of good judgment, all of the physical quantities involved are selected and written down at the beginning, the result will be an equation which is not only dimensionally correct but which also correctly defines the physical phenomenon it is intended to define. The check as to the good judgment lies in the fact that the dimensionless quantities plot as a smooth curve.

In this connection, it might be possible to introduce at the start of a computation some quantity which had no place in the phenomenon and still come out with a dimensionally correct equation. It would be a little difficult, however, to do much with such an equation if it contained some quantity which did not exist in the particular case under consideration and; therefore, could not be measured. However, it would require much less judgment to avoid introducing a superfluous quantity than to insure the inclusion of all of the essential ones.

An example of the possibility of introducing such a quantity might be of interest. Suppose that, in the illustrative example of friction loss in a pipe, there had been added an extra $Q$ having the dimensions $M\frac{L^2}{T^2}$. Incidentally, such a quantity could equally well be energy or a torque, as far as its dimensions are concerned. We would then have

$$\pi_5 = d^a v^b \mu^c X = L^a\left(\frac{L}{T}\right)^b\left(\frac{M}{LT}\right)^c \frac{ML^2}{T^2}$$

$$e + 1 = 0$$
$$a + b - c + 2 = 0$$
$$- b - c - 2 = 0$$

from which

$$a = -2,\ b = -1,\ c = -1$$

and $\pi_5 = \frac{X}{d^2 v \mu}$, which is perfectly correct, dimensionally, but, obviously, has no place in the picture.

In explaining the deduction of the $\pi$-theorem, it was stated that other sets of independent $\pi$'s could be obtained by different choices of the first $k$ quantities in the expression for $\pi$. As an illustration of this, let us consider some physical phenomenon which it will be assumed can be defined by the equation $f\ (d, v, \mu, \rho, g) = 0$. As there are five $Q$'s and three fundamental units, the number of $\pi$'s in any one set will be $5 - 3 = 2$. There are ten possible combinations of the $Q$'s which might be used in deriving the sets of independent $\pi$'s and they are listed below. The first three of the $Q$'s in each set are the ones whose exponents are to be obtained by the method previously shown.

1. $\mu, \rho, g, d$
   $\mu, \rho, g, v$
2. $v, \rho, g, d$
   $v, \rho, g, \mu$
3. $v, \mu, g, d$
   $v, \mu, g, \rho$
4. $v, \mu, \rho, d$
   $v, \mu, \rho, g$
5. $d, \rho, g, v$
   $d, \rho, g, \mu$
6. $d, \mu, g, v$
   $d, \mu, g, \rho$
7. $d, \mu, \rho, v$
   $d, \mu, \rho, g$
8. $d, v, g, \mu$
   $d, v, g, \rho$

9. $d,\ v,\ \rho,\ \mu$
   $d,\ v,\ \rho,\ g$

10. $d,\ v,\ \mu,\ \rho$
    $d,\ v,\ \mu,\ g$

Without reproducing the algebraic computations involved, the sets of independent $\pi$'s resulting from these possible combinations are:

1. $\pi_1 = \dfrac{\rho^{2/3}g^{1/3}d}{\mu^{2/3}}$ , $\pi_2 = \dfrac{\rho^{1/3}v}{\mu^{1/3}g^{1/3}}$

2. $\pi_1 = \dfrac{gd}{v^2}$ , $\pi_2 = \dfrac{g\mu}{v^3\rho}$

3. $\pi_1 = \dfrac{gd}{v^2}$ , $\pi_2 = \dfrac{v^3\rho}{\mu g}$

4. $\pi_1 = \dfrac{v\rho d}{\mu}$ , $\pi_2 = \dfrac{\mu g}{v^3\rho}$

5. $\pi_1 = \dfrac{v}{d^{1/2}g^{1/2}}$ , $\pi_2 = \dfrac{\mu}{d^{3/2}g^{1/2}\rho}$

6. $\pi_1 = \dfrac{v}{d^{1/2}g^{1/2}}$ , $\pi_2 = \dfrac{d^{3/2}g^{1/2}\rho}{\mu}$

7. $\pi_1 = \dfrac{d\rho v}{\mu}$ , $\pi_2 = \dfrac{d^3\rho^2 g}{\mu^2}$

8. No solution.

9. $\pi_1 = \dfrac{\mu}{dv\rho}$ , $\pi_2 = \dfrac{v^2}{dg}$

10. $\pi_1 = \dfrac{dv\rho}{\mu}$ , $\pi_2 = \dfrac{v^2}{dg}$

It sometimes happens that, when a certain choice of $Q$'s is made for obtaining the $\pi$'s, no result can be obtained. This is the case in set No. 8. The equations for obtaining the $\pi$'s would be

$$\pi_1 = d^x v^y g^z \mu = L^x \left(\frac{L}{T}\right)^y \left(\frac{L}{T^2}\right)^z \frac{M}{LT} = M^0L^0T^0$$

$$\pi_2 = d^a v^b g^c \rho = L^a \left(\frac{L}{T}\right)^b \left(\frac{L}{T^2}\right)^c \frac{M}{L^3} = M^0L^0T^0$$

These equations are not valid, as it is obvious that the exponents of $M$

cannot be equated in either one and, therefore, there is no solution for this set of $\pi$'s.

It is to be noted that, wherever $v$, $d$ and $g$ appear among the $Q$'s for a prospective $\pi$, that $\pi$ always turns out to be either Froude's number or a power of it and, whenever $v$, $d$, $\rho$ and $\mu$ thus appear, Reynolds' number, or a power of it, is the result. It is desirable to have Reynolds' and/or Froude's numbers as the dimensionless quantities in the final result when viscous and/or gravity forces enter the problem under consideration. In the present example this could easily have been brought about by choosing combination No. 9 or No. 10 for obtaining the $\pi$'s.

As a more concrete illustration of what happens when, in a given problem, different sets of $\pi$'s are derived, it might be interesting to refer again to the example of the dam with the roller gate and notice that, when we set down $\pi_1$ as $d^a g^b q$ and $\pi_2$ as $d^u g^v h$, there resulted the equation $\dfrac{q}{d^{3/2}g^{1/2}} = \psi\left(\dfrac{h}{d}\right)$. Two other equations can be obtained by letting the $\pi$'s be $d^a q^b g$ and $d^u q^v h$ or by letting them be $h^a g^b q$ and $h^u g^v d$. Without reproducing the algebraic operations, these two equations are, respectively, $\dfrac{d^3 g}{q^2} = \psi\left(\dfrac{h}{d}\right)$ and $\dfrac{q}{h^{3/2}g^{1/2}} = \psi\left(\dfrac{d}{h}\right)$.

In the laboratory work in connection with this particular problem, four runs were made with $d$, the height of the roller-gate opening, kept constant at 4.48 cm., but with $q$ and $h$ varied. Table I gives, for these four runs, the observed values of $q$, in cubic centimeters per second per centimeter of width of channel, and of $h$, in centimeters, and the

TABLE I

| $q$ | $h$ | $\dfrac{q}{d^{3/2}g^{1/2}}$ | $\dfrac{d^3 g}{q^2}$ | $\dfrac{q}{h^{3/2}g^{1/2}}$ | $\dfrac{d}{h}$ | $\dfrac{h}{d}$ |
|---|---|---|---|---|---|---|
| 87.1 | .342 | .291 | 11.64 | 13.91 | 13.12 | .0763 |
| 134.0 | .821 | .498 | 4.93 | 5.75 | 5.46 | .1881 |
| 171.4 | 1.307 | .573 | 3.02 | 3.66 | 3.43 | .2914 |
| 225.5 | 2.211 | .755 | 1.74 | 2.19 | 2.03 | .4930 |

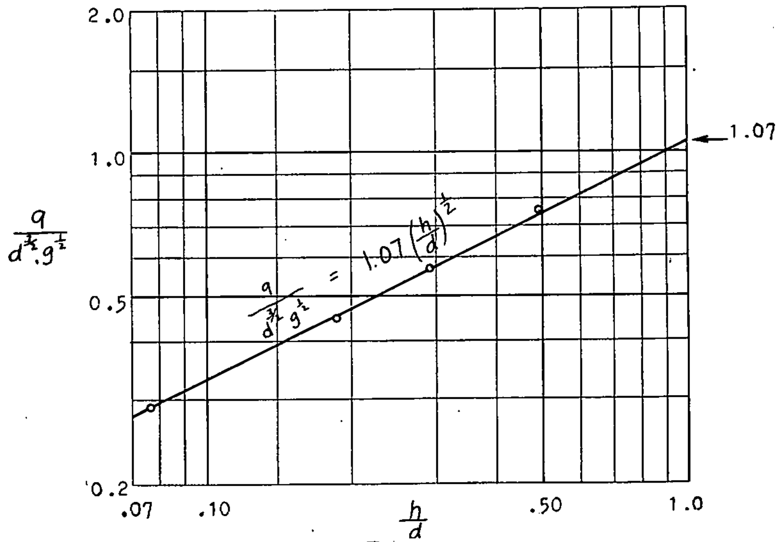
2.0
1.0
0.5
0.2
.07
.10
.50
1.0
1.07
$\frac{q}{d^{3/2} g^{1/2}}$
$\frac{q}{d^{3/2} g^{1/2}} = 1.07 \left(\frac{h}{d}\right)^{1/2}$
$\frac{h}{d}$

FIG. 1

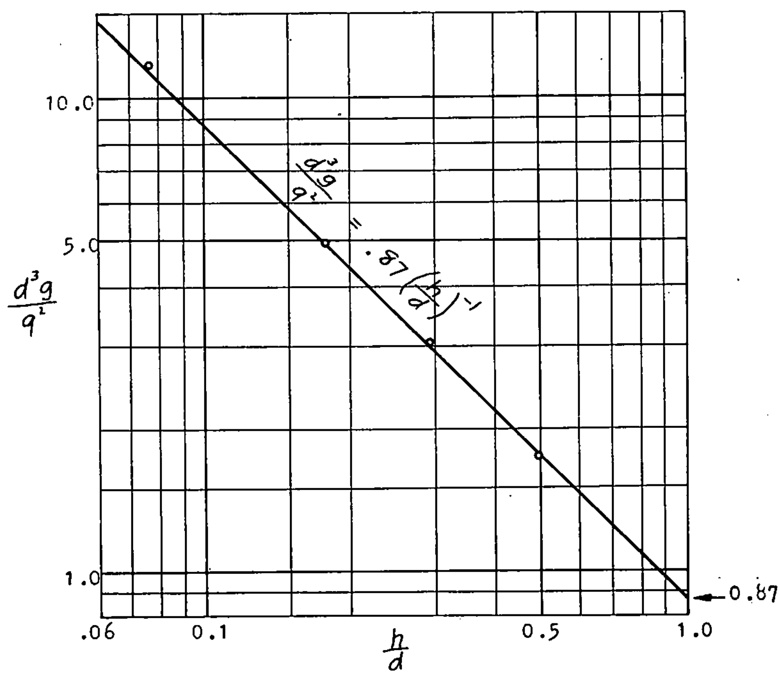
10.0
5.0
1.0
.06
0.1
0.5
1.0
0.87
$\frac{d^3 g}{q^2}$
$\frac{d^3 g}{q^2} = .87 \left(\frac{h}{d}\right)^{-1}$
$\frac{h}{d}$

FIG. 2

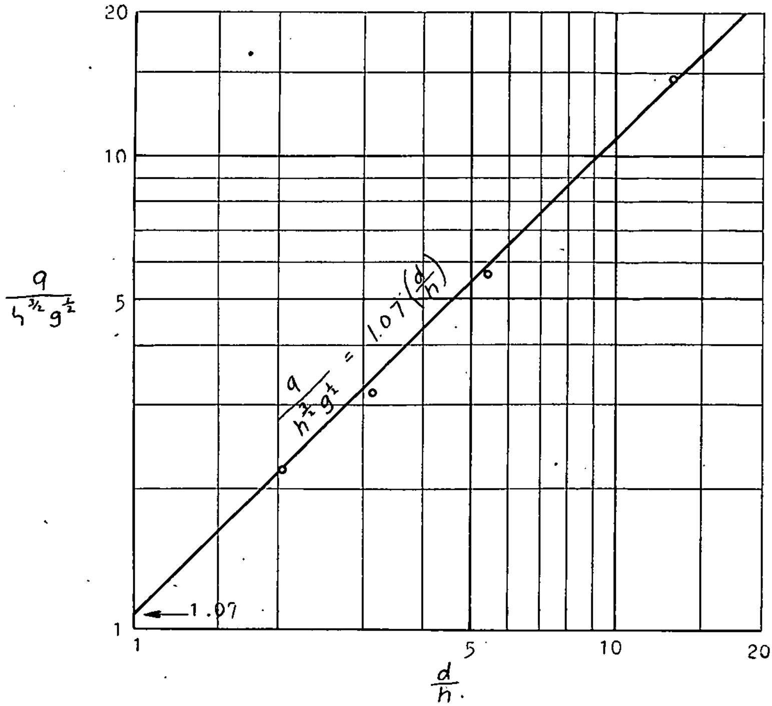


FIG. 3

computed values of the five dimensionless quantities involved in these three equations. Figs. 1, 2 and 3 are the logarithmic plots of the data pertaining to the respective equations, from which it is seen that the equations take the form:

$$\frac{q}{d^{3/2}g^{1/2}} = 1.07 \left(\frac{h}{d}\right)^{1/2}$$

$$\frac{d^3 g}{q^2} = 87 \left(\frac{h}{d}\right)^{-1}$$

$$\frac{q}{h^{3/2}g^{1/2}} = 1.07 \left(\frac{d}{h}\right)$$

At first glance, these three might appear to be different equations, but they all reduce to $q = .76\, d\sqrt{2\, g\, h}$, which illustrates the fact that, in any given problem, any one of the possible sets of $\pi$'s may be selected without affecting the final result.

The foregoing observations have been set down with the object of clearing up certain questions which have arisen in a study of dimensional analysis. If they be as helpful to others in a better understanding of this most useful tool as they have been to me, they will have served their purpose.

# BEAMS OF VARIABLE MOMENT OF INERTIA

## The Computation of Fixed End Moments, Sidesway Moments, Stiffnesses and Carry-Over Factors

BY JOHN B. WILBUR*

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1. Synopsis: In this paper procedures are suggested for a simplified method based on tabular values and equations for the determination of fixed end moments, sidesway moments, stiffnesses and carry-over factors for use in moment distribution solution of rigid frames. This method permits one to take into account variation of moment of inertia of the members involved. It is applicable to unsymmetrical members, but can be simplified for symmetrical members, and simplified still further for symmetrical members acted upon by summetrical loads. As a corollary of the main purpose of this paper, expressions are also derived which permit taking variable moment of inertia into account in the slope deflection equations.

The development of these procedures is based on the moment area theorems. No new theories are introduced, but it is believed that the proposed method will be useful because of the fact that it is so arranged as to be readily used by one not familiar with the moment area theorems themselves.

The results obtained by this method will be slightly approximate because the beam is assumed to be divided into ten equal parts, and the bending moment divided by the moment of inertia assumed to vary linearly through each of the ten sections thus obtained. It is believed however that the error introduced by this approximation will be negligible unless the variation in moment of inertia is unusual. The results could be made more accurate by dividing the beam into a greater number of parts, but it is felt that this additional accuracy would probably be more than offset by the extra work involved in its application.

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TABLE I - FORMULAS FOR VARIOUS FUNCTIONS FOR BEAMS OF VARIABLE I

| | Function | Unsymmetrical Beam - All Loads | | Symmetrical Beam Unsymmetrical Loads | | Symmetrical Beam Symmetrical Loads | |
|---|---|---|---|---|---|---|---|
| | | Left End | Right End | Left End | Right End | Left End | Right End |
| For Moment Distribution | Fixed End Moment | $\frac{10(A_2C_3-A_3C_2)}{(B_3C_2-B_2C_3)}$ | $\frac{10(A_2B_3-A_3B_2)}{(B_3C_2-B_2C_3)}$ | $\frac{10(A_2B_2-A_3C_2)}{(C_2^2-B_2^2)}$ | $\frac{10(A_2C_2-A_3B_2)}{(C_2^2-B_2^2)}$ | $-\frac{10\ A_2}{(C_2+B_2)}$ | $+\frac{10\ A_2}{(C_2+B_2)}$ |
| | Sidesway Moment | $\frac{(C_2+C_3)}{(B_2C_3-B_3C_2)}1000\frac{dE}{L^2}$ | $\frac{(B_3+B_2)}{(B_2C_3-B_3C_2)}1000\frac{dE}{L^2}$ | $\frac{1000\ dE}{(B_2-C_2)L^2}$ | Same as Left End | Same as for unsymmetrical loads | |
| | Carry Over Factor | (Left to Right) $\frac{B_2}{C_2}$ | (Right to Left) $\frac{C_3}{B_3}$ | (Left to Right) $\frac{B_2}{C_2}$ | (Right to Left) Same as Left to Right | Same as for unsymmetrical loads | |
| | Stiffness | $\frac{C_2}{(B_1C_2-B_2C_1)}\ 100\ \frac{E}{L}$ | $\frac{B_3}{(B_3C_1-B_1C_3)}\ 100\ \frac{E}{L}$ | $\frac{C_2}{B_1(C_2-B_2)}100\frac{E}{L}$ | Same as Left End | Same as for unsymmetrical loads | |
| For Slope Deflection | Coefficient to $\theta_A$ | $\alpha = \frac{C_2}{(B_1C_2-B_2C_1)}\ 100\ \frac{E}{L}$ | $\alpha' = \frac{B_2}{(B_1C_2-B_2C_1)}\ 100\ \frac{E}{L}$ | $\alpha = \frac{C_2}{B_1(C_2-B_2)}100\frac{E}{L}$ | $\alpha' = \frac{B_2}{B_1(C_2-B_2)}100\frac{E}{L}$ | Same as for unsymmetrical loads | |
| | Coefficient to $\theta_B$ | $\beta = \frac{C_3}{(B_3C_1-B_1C_3)}\ 100\ \frac{E}{L}$ | $\beta' = \frac{B_3}{(B_3C_1-B_1C_3)}\ 100\ \frac{E}{L}$ | $\beta = \frac{B_2}{B_1(C_2-B_2)}100\frac{E}{L}$ | $\beta' = \frac{C_2}{B_1(C_2-B_2)}100\frac{E}{L}$ | Same as for unsymmetrical loads | |
| | Coefficient to R | $\gamma = \frac{(C_2+C_3)}{(B_2C_3-B_3C_2)}1000\ \frac{E}{L}$ | $\gamma' = \frac{(B_3+B_2)}{(B_2C_3-B_3C_2)}1000\ \frac{E}{L}$ | $\gamma = \frac{1000}{(B_2-C_2)}\ \frac{E}{L}$ | $\gamma' = \frac{1000}{(B_2-C_2)}\ \frac{E}{L}$ | Same as for unsymmetrical loads | |
| | Fixed End Moment | $\frac{10(A_2C_3-A_3C_2)}{B_3C_2-B_2C_3}$ | $\frac{10(A_2B_3-A_3B_2)}{B_3C_2-B_2C_3}$ | $\frac{10(A_2B_2-A_3C_2)}{C_2^2-B_2^2}$ | $\frac{10(A_2C_2-A_3B_2)}{C_2^2-B_2^2}$ | $-\frac{10\ A_2}{(C_2+B_2)}$ | $+\frac{10\ A_2}{(C_2+B_2)}$ |

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J.B. Wilbur 4/21/42

In the body of this paper, equations will be given and procedure described without supporting derivations. These derivations are included in Appendices to the paper, as is also a summary of notation symbols adopted.

2. **Description of Table of Formulas:** In Table I, formulas are given which express various functions in terms of eight symbols, namely, $A_2$, $A_3$, $B_1$, $B_2$, $B_3$, $C_1$, $C_2$, and $C_3$, From these formulas, if values of those of the eight symbols required for the formula for the particular function required have been previously evaluated, the following quantities may be determined:

| For use in moment distribution: | For use in slope deflection: |
|---|---|
| Fixed End Moments; | Coefficient to $\theta_A$; |
| Sidesway Moments; | Coefficient to $\theta_B$; |
| Carry-Over Factors; | Coefficient to $R$; |
| Stiffnesses. | Fixed End Moments. |

These quantities may be obtained for either end of the beam under consideration. It should be noted that for an unsymmetrical beam, all the above functions have different values for the two ends.

Separate formulas, which are in general simpler, are given for the case of a symmetrical beam acted upon by an unsymmetrical load. For these conditions all of the foregoing functions for use in moment distribution except fixed end moments have the same value at each end of the beam.

If the loading also is symmetrical, the formulas for fixed end moments simplify still further and have the same form for each end of the beam. Formulas for these conditions are also given in the table.

The formulas for the coefficients to $\theta_A$, $\theta_B$ and $R$ and for fixed end moments, all for use in the slope deflection equations, are to be substituted in the following equations:

For the $A$ or left end of the beam $AB$:

$$M_{AB} = \alpha\theta_A + \beta\theta_B + \gamma R + \text{Fixed End Moment.}$$

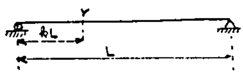
For the $B$ or right end of the beam $AB$:

$$M_{BA} = \alpha'\theta_A + \beta'\theta_B + \gamma' R' + \text{Fixed End Moment.}$$

3. **Evaluation of Eight Symbols $A_2$, $A_3$, $B_1$, $B_2$, $B_3$, $C_1$, $C_2$ and $C_3$:** These symbols may be evaluated by means of the values given in Table II. The procedure is as follows:

TABLE II

$\frac{1}{I}$ and $\frac{M_B}{I}$ computed at this point




| k | $A_2$ | $A_3$ | $B_1$ | $B_2$ | $B_3$ | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.167 | 4.833 | 5.000 | 1.667 | 48.333 | 0.000 | 0.000 | 0.000 |
| 0.1 | 1.000 | 9.000 | 9.000 | 9.000 | 81.000 | 1.000 | 1.000 | 9.000 |
| 0.2 | 2.000 | 8.000 | 8.000 | 16.000 | 64.000 | 2.000 | 4.000 | 16.000 |
| 0.3 | 3.000 | 7.000 | 7.000 | 21.000 | 49.000 | 3.000 | 9.000 | 21.000 |
| 0.4 | 4.000 | 6.000 | 6.000 | 24.000 | 36.000 | 4.000 | 16.000 | 24.000 |
| 0.5 | 5.000 | 5.000 | 5.000 | 25.000 | 25.000 | 5.000 | 25.000 | 25.000 |
| 0.6 | 6.000 | 4.000 | 4.000 | 24.000 | 16.000 | 6.000 | 36.000 | 24.000 |
| 0.7 | 7.000 | 3.000 | 3.000 | 21.000 | 9.000 | 7.000 | 49.000 | 21.000 |
| 0.8 | 8.000 | 2.000 | 2.000 | 16.000 | 4.000 | 8.000 | 64.000 | 16.000 |
| 0.9 | 9.000 | 1.000 | 1.000 | 9.000 | 1.000 | 9.000 | 81.000 | 9.000 |
| 1.0 | 4.833 | 0.167 | 0.000 | 0.000 | 0.000 | 5.000 | 48.333 | 1.667 |
| Multiplier | $\frac{\Sigma(\ )M_B}{I} = A_2$ | $\frac{\Sigma(\ )M_B}{I} = A_3$ | $\frac{\Sigma(\ )}{I} = B_1$ | $\frac{\Sigma(\ )}{I} = B_2$ | $\frac{\Sigma(\ )}{I} = B_3$ | $\frac{\Sigma(\ )}{I} = C_1$ | $\frac{\Sigma(\ )}{I} = C_2$ | $\frac{\Sigma(\ )}{I} = C_3$ |

*a.* Determine value of $\frac{1}{I}$ at each tenth point along span ($k = 0.0$, $k = 0.1$, $k = 0.2$, etc., as shown on sketch at top of Table II).

*b.* Determine value of $M_B$, the moment due to the applied transverse loads considering the beam as end supported, at each tenth point along the span.

*c.* Determine value of $\frac{M_B}{I}$ at each tenth point along the span.

(Note that values of $M_B$ and $\frac{M_B}{I}$ need be evaluated only when $A_2$

and $A_3$ are to be determined, which is necessary only when fixed end moments are to be computed.)

To evaluate $A_2$, multiply each number in the column headed $A_2$, by the value of $\frac{M_B}{I}$ corresponding to the value of $k$ opposite the number in the table. The sum of the products thus obtained is $A_2$.

Values of $A_3$ are obtained by an identical procedure, using values in Table II from the column headed $A_3$.

To evaluate $B_1$ multiply each number in the column headed $B_1$ by the value of $\frac{1}{I}$ corresponding to the value of $k$ opposite the number in the table. The sum of the products thus obtained is $B_1$.

Values of $B_2$, $B_3$, $C_1$, $C_2$ and $C_3$ are obtained by identical procedures, using values in Table II from the column headed by the symbol desired.

4. **Illustrative Problem:** Determine the Fixed End Moments (*F.E.M.*), Sidesway Moment (*S.M.*), Carry Over Factor (*C.O.F.*) and Stiffness ($K$) for the beam shown below, with a single concentrated load of 1,000 lbs. acting at mid span.

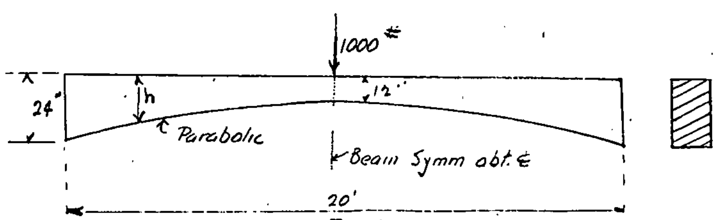


Fig. 1

Relative values of $I$ will be used. This leads to true values of *F.E.M.* and *C.O.F.*, and relative values of *S.M.* and $K$. Relative values of these latter two functions suffice for a moment distribution solution so long as the same proportional factors are used throughout a structure.

*A*. Preliminary data:

| $k$ | $h$ | $h^3 = \text{rel.}\ I$ | $\frac{10{,}000}{h^3} = \text{rel.}\ \frac{1}{I}$ | $\frac{M_B}{(\text{Kip ft.})}$ | $\frac{M_B}{\text{rel.}\ I}$ |
|---|---|---|---|---|---|
| 0.0,1.0 | $12(1+25/25) = 24.0$ | 13,850 | 0.722 | 0.000 | 0 |
| 0.1,0.9 | $12(1+16/25) = 19.7$ | 7,650 | 1.308 | 1.000 | 1.31 |
| 0.2,0.8 | $12(1+\ 9/25) = 16.3$ | 4,330 | 2.310 | 2.000 | 4.62 |
| 0.3,0.7 | $12(1+\ 4/25) = 13.9$ | 2,690 | 3.720 | 3.000 | 11.16 |
| 0.4,0.6 | $12(1+\ 1/25) = 12.5$ | 1,950 | 5.130 | 4.000 | 20.52 |
| 0.5 | $12(1+\ 0/25) = 12.0$ | 1,730 | 5.780 | 5.000 | 28.90 |

*B*. Formulas to be used: (From Table I: Symmetrical Beam Symmetrically Loaded):

$$F.E.M. = \frac{10\,A_2}{C_2 + B_2};\quad S.M. = \frac{1{,}000\,dE}{(B_2 - C_2)L^2};$$

$$C.O.F. = \frac{B_2}{C_2};\quad K = \frac{C_2}{B_1(C_2 - B_2)}\,100\,\frac{E}{L}.$$

*C*. Determination of $A_2$, $B_1$, $B_2$ and $C_2$: (Based on Table II):

| $k$ | $A_2$ | $B_1$ | $B_2$ | $C_2$ |
|---|---|---|---|---|
| 0.0 | 0 | 3.6 | 1.2 | 0 |
| 0.1 | 1.3 | 11.7 | 11.7 | 1.3 |
| 0.2 | 9.2 | 18.5 | 37.0 | 9.3 |
| 0.3 | 33.5 | 26.0 | 78.2 | 33.5 |
| 0.4 | 82.2 | 30.7 | 123.0 | 82.0 |
| 0.5 | 144.5 | 28.9 | 144.5 | 144.5 |
| 0.6 | 123.2 | 20.5 | 123.0 | 184.5 |
| 0.7 | 78.2 | 11.1 | 78.2 | 182.5 |
| 0.8 | 36.9 | 4.6 | 37.0 | 148.0 |
| 0.9 | 11.7 | 1.3 | 11.7 | 105.8 |
| 1.0 | 0 | 0 | 0 | 34.8 |

$A_2 = 520.8;\ B_1 = 157.1;\ B_2 = 645.6;\ C_2 = 926.2$

*D.* Determination of Required Functions:

$$F.E.M. = \frac{10(520.8)}{645.6 + 926.2} = 3.36 \text{ Kip ft.}$$

$$S.M. = \frac{1{,}000\,dE}{645.6 - 926.2} \cdot \frac{1}{L^2} = -3.56\,\frac{dE}{L^2} \text{ (relative value)}$$

$$C.O.F. = \frac{645.6}{926.2} = 0.698$$

$$K = \frac{926.2}{151.7(926.2 - 645.6)}\,100\,\frac{E}{L} = 2.10\,\frac{E}{L} = 0.105E$$

(relative value)

**Appendix *A*: Areas under $\frac{M}{I}$ Curves:** Assume that the beam under consideration be divided into ten parts of equal length, and that the $\frac{M}{I}$ curve is linear for each segment, as shown in Figure 2.

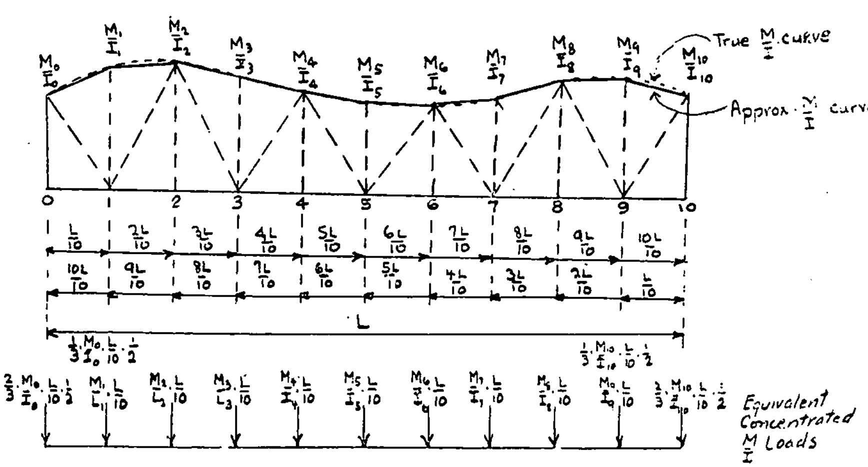



FIG. 2

If $M_0$, $M_1$, etc., are values of $M_B$, then

$$a_1 = \frac{L}{10}\left(\frac{1}{2}\,\frac{M_0}{I_0} + \frac{M_1}{I_1} + \frac{M_2}{I_2} + \frac{M_3}{I_3} + \frac{M_4}{I_4} + \frac{M_5}{I_5} + \frac{M_6}{I_6} + \frac{M_7}{I_7}\right.$$

$$+\frac{M_8}{I_8}+\frac{M_9}{I_9}+\frac{1}{2}\,\frac{M_{10}}{I_{10}}\Big) \tag{a}$$

If $M_0$, $M_1$, etc., are values of moment due to $M_L$, then, since

$$M_0=10\frac{M_L}{10};\ M_1=\frac{9}{10}M_L;\ M_2=\frac{8}{10}M_L,\ \text{etc.,}$$

$$b_1=\frac{M_LL}{100}\left(\frac{5}{I_0}+\frac{9}{I_1}+\frac{8}{I_2}+\frac{7}{I_3}+\frac{6}{I_4}+\frac{5}{I_5}+\frac{4}{I_6}+\frac{3}{I_7}+\frac{2}{I_8}+\frac{1}{I_9}\right) \tag{b}$$

If $M_0$, $M_1$, etc., are values of moment due to $M_R$, then, since

$$M_0=0;\ M_1=\frac{M_R}{10};\ M_2=\frac{2M_R}{10},\ \text{etc.}$$

$$c_1=\frac{M_RL}{100}\left(\frac{1}{I_1}+\frac{2}{I_2}+\frac{3}{I_3}+\frac{4}{I_4}+\frac{5}{I_5}+\frac{6}{I_6}+\frac{7}{I_7}+\frac{8}{I_8}+\frac{9}{I_9}+\frac{5}{I_{10}}\right) \tag{c}$$

Using the symbols $A_1$, $B_1$ and $C_1$ as defined under notation:

$$a_1=\frac{L}{10}A_1 \tag{d}$$

$$b_1=\frac{M_LL}{100}B_1 \tag{e}$$

$$c_1=\frac{M_RL}{100}C_1 \tag{f}$$

**Appendix $B$: Moments of Areas under $\frac{M}{I}$ Curves about Left End of Beam:**

If $M_0$, $M_1$, etc., are values of $M_B$, then

$$a_2=\frac{1}{3}\cdot\frac{M_0}{I_0}\cdot\frac{L}{10}\cdot\frac{1}{2}\cdot\frac{L}{10}+\frac{M_1}{I_1}\cdot\frac{L}{10}\cdot\frac{L}{10}+\frac{M_2}{I_2}\cdot\frac{L}{10}\cdot\frac{2L}{10}+\frac{M_3}{L_3}\cdot$$

$$\frac{L}{10}\cdot\frac{3L}{10}+\frac{M_4}{I_4}\cdot\frac{L}{10}\cdot\frac{4L}{10}+\frac{M_5}{I_5}\cdot\frac{L}{10}\cdot\frac{5L}{10}+\frac{M_6}{I_6}\cdot\frac{L}{10}\cdot\frac{6L}{10}+\frac{M_7}{I_7}\cdot$$

$$\frac{L}{10}\cdot\frac{7L}{10}+\frac{M_8}{I_8}\cdot\frac{L}{10}\cdot\frac{8L}{10}+\frac{M_9}{I_9}\cdot\frac{L}{10}\cdot\frac{9L}{10}+\frac{1}{3}\cdot\frac{M_{10}}{L_{10}}\cdot\frac{L}{10}\cdot\frac{1}{2}\cdot$$

$$\frac{9L}{10}+\frac{2}{3}\cdot\frac{M_{10}}{L_{10}}\cdot\frac{L}{10}\cdot\frac{1}{2}\cdot\frac{10L}{10}$$

$$=\frac{L^2}{100}\left(\frac{1}{6}\frac{M_0}{I_0}+\frac{M_1}{I_1}+\frac{2M_2}{I_2}+\frac{3M_3}{I_3}+\frac{4M_4}{I_4}+\frac{5M_5}{I_5}+\frac{6M_6}{I_6}+\right.$$

$$\left.\frac{7M_7}{I_7}+\frac{8M_8}{I_8}+\frac{9M_9}{I_9}+4.8\dot{3}\,\frac{M_{10}}{I_{10}}\right) \qquad (a)$$

If $M_0$, $M_1$, etc., are values of moment due to $M_L$, then since

$$M_0=\frac{10M_L}{10};\ M_1=\frac{9M_L}{10},\ \text{etc.}$$

$$b_2=\frac{M_LL^2}{1{,}000}\left(\frac{1.\dot{6}}{I_0}+\frac{9}{I_1}+\frac{16}{I_2}+\frac{21}{I_3}+\frac{24}{I_4}+\frac{25}{I_5}+\frac{24}{I_6}+\frac{21}{I_7}+\frac{16}{I_8}\right.$$

$$\left.+\frac{9}{I_9}\right) \qquad (b)$$

If $M_0$, $M_1$, etc., are values of moment due to $M_R$, then since

$$M_0=0,\ M_1=\frac{M_R}{10},\ M_2=\frac{2M_R}{10}\ \text{etc.,}$$

$$c_2=\frac{M_RL^2}{1{,}000}\left(\frac{1}{I_1}+\frac{4}{I_2}+\frac{9}{I_3}+\frac{16}{I_4}+\frac{25}{I_5}+\frac{36}{I_6}+\frac{49}{I_7}+\frac{64}{I_8}+\frac{81}{I_9}+\right.$$

$$\left.\frac{48.\dot{3}}{I_{10}}\right) \qquad (c)$$

Using the symbols $A_2$, $B_2$ and $C_2$ as defined under notation:

$$a_2 = \frac{L^2}{100} A_2 \qquad (d)$$

$$b_2 = \frac{M_L L^2}{1,000} B_2 \qquad (e)$$

$$c_2 = \frac{M_R L^2}{1,000} C_2 \qquad (f)$$

**Appendix C: Moments of Areas under $\frac{M}{I}$ curves about Right End of Beam:**

If $M_0$, $M_1$, etc., are values of $M_B$, then

$$a_3 = \frac{2}{3} \cdot \frac{M_0}{I_0} \cdot \frac{L}{10} \cdot \frac{1}{2} \cdot \frac{10L}{10} + \frac{1}{3} \cdot \frac{M_0}{I_0} \cdot \frac{L}{10} \cdot \frac{1}{2} \cdot \frac{9L}{10} + \frac{M_1}{I_1} \cdot \frac{L}{10} \cdot \frac{9L}{10} + \frac{M_2}{I_2} \cdot \frac{L}{10} \cdot \frac{8L}{10} + \frac{M_3}{I_3} \cdot \frac{L}{10} \cdot \frac{7L}{10} + \frac{M_4}{I_4} \cdot \frac{L}{10} \cdot \frac{6L}{10} + \frac{M_5}{I_5} \cdot \frac{L}{10} \cdot \frac{5L}{10} + \frac{M_6}{I_6} \cdot \frac{L}{10} \cdot \frac{4L}{10} + \frac{M_7}{I_7} \cdot \frac{L}{10} \cdot \frac{3L}{10} + \frac{M_8}{L_8} \cdot \frac{L}{10} \cdot \frac{2L}{10} + \frac{M_9}{L_9} \cdot \frac{L}{10} \cdot \frac{L}{10} + \frac{1}{3} \cdot \frac{M_{10}}{L_{10}} \cdot \frac{L}{10} \cdot \frac{1}{2} \cdot \frac{L}{10}$$

$$= \frac{L^2}{100}\left(4.83 \frac{M_0}{I_0} + \frac{9M_1}{I_1} + \frac{8M_2}{I_2} + \frac{7M_3}{I_3} + \frac{6M_4}{I_4} + \frac{5M_5}{I_5} + \frac{4M_6}{I_6} + \frac{3M_7}{I_7} + \frac{2M_8}{I_8} + \frac{M_9}{I_9} + \frac{1}{6}\frac{M_{10}}{I_{10}}\right) \qquad (a)$$

If $M_0$, $M_1$, etc., are values of moment due to $M_L$, then since

$$M_0 = \frac{10M_L}{10}, \quad M_1 = \frac{9M_L}{10}, \text{ etc.}$$

$$b_3 = \frac{M_L L^2}{1{,}000}\left(\frac{48.3}{I_0} + \frac{81}{I_1} + \frac{64}{I_2} + \frac{49}{I_3} + \frac{36}{I_4} + \frac{25}{I_5} + \frac{16}{I_6} + \frac{9}{I_7} + \frac{4}{I_8} + \frac{1}{I_9}\right) \qquad (b)$$

If $M_0$, $M_1$, etc., are values of moment due to $M_R$, then since $M_0 = 0$; $M_1 = \frac{M_R}{10}$; $M_2 = \frac{2M_R}{10}$, etc.,

$$c_3 = \frac{M_R L^2}{1{,}000}\left(\frac{9}{I_1} + \frac{16}{I_2} + \frac{21}{I_3} + \frac{24}{I_4} + \frac{25}{I_5} + \frac{24}{I_6} + \frac{21}{I_7} + \frac{16}{I_8} + \frac{9}{I_9} + \frac{1.6}{I_{10}}\right) \qquad (c)$$

Using symbols $A_3$, $B_3$ and $C_3$ as defined under notation,

$$a_3 = \frac{L^2}{100} A_3 \qquad (d)$$

$$b_3 = \frac{M_L L^2}{1{,}000} B_3 \qquad (e)$$

$$c_3 = \frac{M_R L^2}{1{,}000} C_3 \qquad (f)$$

**Appendix *D*: Equations for Fixed End Moments:** Since at each end of a fixed end beam both slope and deflection are zero, the deflection of each end from the tangent at the other end is zero. This permits the writing of two equations based on the second moment area theorem. These two equations involve the moments at the ends of the beam as unknowns: the simultaneous solution of these equations leads to the fixed end moments.

Taking moments of the areas under the $\frac{M}{I}$ curves about the left and right end of the span respectively

$$a_2 = b_2 + c_2$$
$$a_3 = b_3 + c_3$$

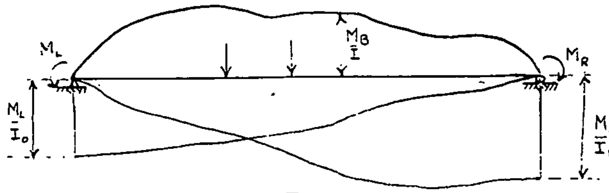



FIG. 3

From the relations developed in Appendices $B$ and $C$, these equations may be written as

$$\frac{L^2}{100}A_2 = \frac{M_L L^2}{1{,}000}B_2 + \frac{M_R L^2}{1{,}000}C_2$$
$$\frac{L^2}{100}A_3 = \frac{M_L L^2}{1{,}000}B_3 + \frac{M_R L^2}{1{,}000}C_3$$

which reduce to

$$10\,A_2 = B_2 M_L + C_2 M_R$$
$$10\,A_3 = B_3 M_L + C_3 M_R$$

The simultaneous solution of these equations leads to

$$M_L = \frac{10(A_2C_3 - A_3C_2)}{B_3C_2 - B_2C_3} \qquad (a)$$

$$M_R = \frac{10(A_2B_3 - A_3B_2)}{B_3C_2 - B_2C_3} \qquad (b)$$

The signs in equation $(a)$ have been changed by multiplying the denominator by —1. With this change, positive values of $M_L$ and $M_R$ as obtained from equations $(a)$ and $(b)$ respectively, act clockwise on the ends of the member.

If a beam is symmetrical, $C_3 = B_2$ and $B_3 = C_2$, whence equations $(a)$ and $(b)$ become

$$M_L = \frac{10(A_2B_2 - A_3C_2)}{C_2^2 - B_2^2} \qquad (c)$$

$$M_R = \frac{10(A_2C_2 - A_3B_2)}{C_2^2 - B_2^2} \qquad (d)$$

If the loading as well as the beam is symmetrical, equations $(c)$ $(d)$ become, since $A_3 = A_2$,

$$M_L = -\frac{10A_2}{C_2 + B_2} \tag{e}$$

$$M_R = +\frac{10A_2}{C_2 + B_2} \tag{f}$$

**Appendix *E*: Equations for Sidesway Moments:** If a member is subjected to sidesway only, its ends undergo no rotation. Thus the deflection of either end of the member, measured relative to the tangent at the other end, is given by *d*. This permits the writing of two equations, based on the second moment area theorem.

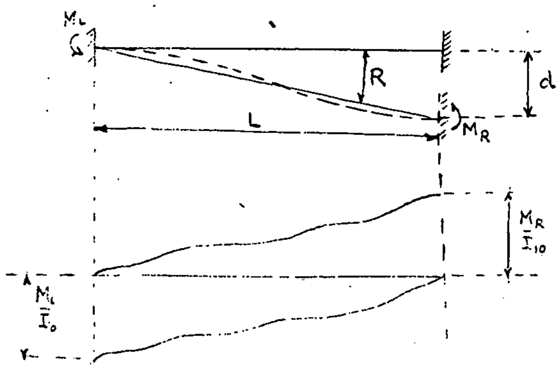


FIG. 4

$$\frac{1}{E}(b_2 - c_2) = +d$$

$$\frac{1}{E}(b_3 - c_3) = -d$$

From the relations of Appendices *B* and *C*, these equations become

$$\frac{M_L L^2}{1{,}000} B_2 - \frac{M_R L^2}{1{,}000} C_2 = + dE$$

$$\frac{M_L L^2}{1{,}000} B_3 - \frac{M_R L^2}{1{,}000} C_3 = - dE$$

Hence

$$B_2 M_L - C_2 M_R = + \frac{1{,}000\, dE}{L^2}$$

$$B_3 M_L - C_3 M_R = - \frac{1{,}000\, dE}{L^2}$$

The simultaneous solution of these equations leads to

$$M_L = \frac{(C_2 + C_3)}{(B_2C_3 - B_3C_2)} \quad \frac{1{,}000\, dE}{L^2} \tag{a}$$

$$M_R = \frac{(B_3 + B_2)}{(B_2C_3 - B_3C_2)} \quad \frac{1{,}000\, dE}{L^2} \tag{b}$$

If $d$ is taken as plus when $\frac{d}{L} = R$ represents a clockwise rotation of the chord joining the ends of the member, positive values of $M_L$ and $M_R$ from equations $(a)$ and $(b)$ respectively, act clockwise on the ends of the member.

If the beam is symmetrical, $C_3 = B_2$ and $B_3 = C_2$, whence

$$M_L = M_R = + \frac{1{,}000\, dE}{(B_2 - C_2)L^2} \tag{c}$$

**Appendix *F*: Carry-Over Factors:** The carry-over factor is the ratio between the moment developed at the far end of a beam which is fixed, and the moment applied at the near end of the beam which is hinged. Since the slope at the far end is zero, the deflection of the near end from the tangent at the far end is zero. This permits

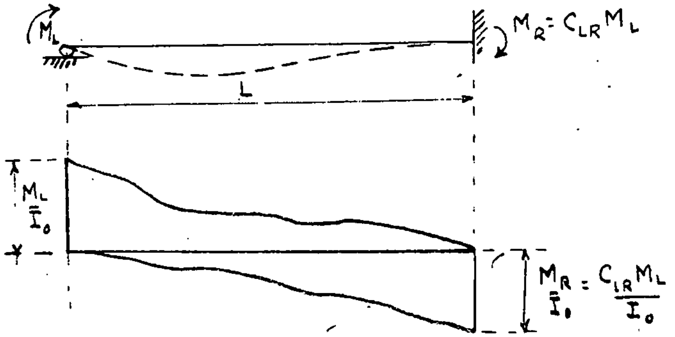



FIG. 5

writing an equation based on the second moment area theorem. If we desire to obtain $C_{LR}$, the carry-over factor from left to right,

$$\therefore b_2 = c_2$$

Using the relations of Appendices $B$ and $C$, and noting that $M_R = C_{LR}M_L$

$$\frac{M_L L^2}{1{,}000} B_2 = \frac{M_R L^2}{1{,}000} C_2 = \frac{C_{LR} M_L L^2}{1{,}000} C_2$$

$$\therefore C_{LR} = \frac{B_2}{C_2} \tag{a}$$

Similarly if we desire to obtain $C_{RL}$, the carry-over factor from right to left

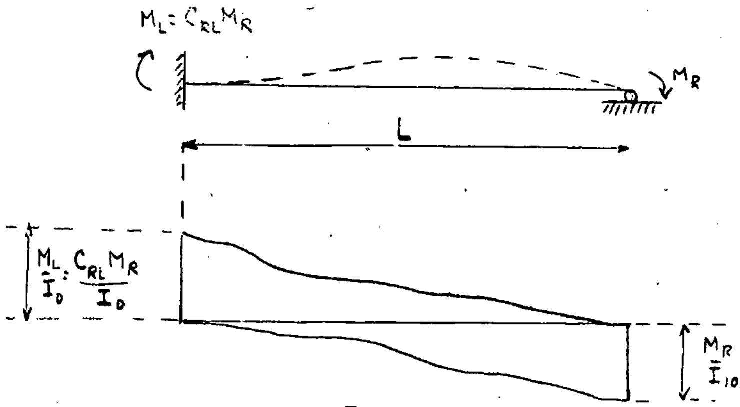



FIG. 6

$$b_3 = c_3$$

$$\frac{M_L L^2}{1{,}000} B_3 = \frac{C_{RL} M_R L^2}{1{,}000} B_3 = \frac{M_R L^2}{1{,}000} C_3$$

$$\therefore C_{RL} = \frac{C_3}{B_3} \tag{b}$$

If the beam is symmetrical, $C_3 = B_2$ and $B_3 = C_2$ whence

$$C_{RL} = C_{LR} = C = \frac{B_2}{C_2} \tag{c}$$

**Appendix G: Stiffness:** The stiffness of a beam is defined as the moment required to produce a unit rotation at a hinged end of the beam when the other end is fixed. By the first moment area theorem, it follows that for this case, the net area under the $\dfrac{M}{EI}$ curve is 1, whence the net area under the $\dfrac{M}{I}$ curve is $E$. Let it be desired to compute $K_L$ the stiffness of the left end of a beam.

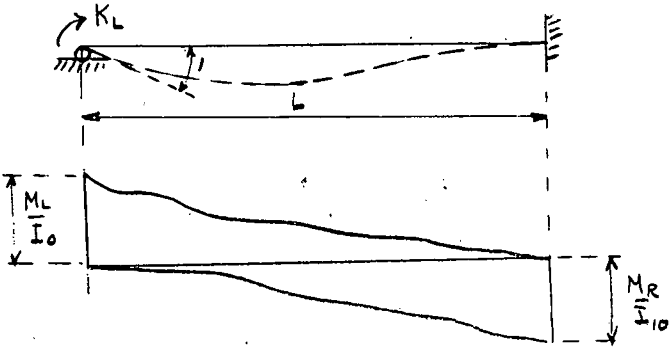



FIG. 7

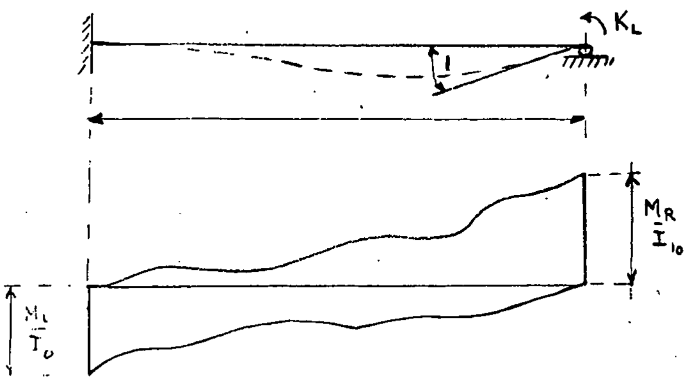



FIG. 8

$$b_1 - c_1 = E$$

$$\therefore \frac{M_L L}{100} B_1 - \frac{M_R L}{100} C_1 = E; \text{ but } M_L = K_L \text{ and } M_R = C_{LR} K_L = \frac{B_2}{C_2} K_L$$

$$\therefore \frac{K_L L}{100} B_1 - \frac{B_2 K_L}{C_2} \frac{L}{100} C_1 = E$$

$$\therefore K_L\left(B_1 + B_2 \frac{C_1}{C_2}\right) = 100 \frac{E}{L}$$

$$\therefore K_L = 100 \frac{E}{L}\left(\frac{C_2}{B_1C_2 - B_2C_1}\right) \qquad (a)$$

Similarly, if we desire to obtain $K_R$, the stiffness of the right end of the beam,

$$c_1 - b_1 = E$$

$$\therefore \frac{M_RL}{100} C_1 - \frac{M_LL}{100} B_1 = E; \text{ but } M_R = K_R \text{ and } M_L = C_{RL}K_R = \frac{C_3}{B_3} K_R$$

$$\therefore \frac{K_RL}{100} C_1 - \frac{C_3}{B_3} \frac{K_RL}{100} B_1 = E$$

$$\therefore K_R\left(C_1 - B_1 \frac{C_3}{B_3}\right) = 100 \frac{E}{L}$$

$$\therefore K_R = 100 \frac{E}{L}\left(\frac{B_3}{B_3C_1 - B_1C_3}\right) \qquad (b)$$

For a symmetrical beam $C_3 = B_2$; $B_3 = C_2$ and $B_1 = C_1$ whence

$$K_L = K_R = 100 \frac{E}{L}\left[\frac{C_2}{B_1(C_2 - B_2)}\right] \qquad (c)$$

**Appendix *H*: Slope Deflection Equations:** In the slope deflection equations the moment at each end of a beam $AB$ is expressed by superimposing the effects of $\theta_A$, the slope at end $A$; $\theta_B$, the slope at end $B$; $R$, the angular rotation of the chord, and $F$, the fixed end moment for the transverse loads. If $A$ is the left end of beam $AB$, it follows by the definition of stiffness, that $M_{AB}$, the moment at $A$ due to $\theta_A$ equals the stiffness at the $A$ end multiplied by $\theta_A$, or

$$M_{AB} = 100 \frac{E}{L}\left(\frac{C_2}{B_1C_2 - B_2C_1}\right) \theta_A \qquad (a)$$

Due to $\theta_B$, $M_{AB}$ equals the stiffness at the $B$ end multiplied by $\theta_B$, multiplied by the carry-over factor from $B$ to $A$, or

$$M_{AB} = 100 \frac{E}{L}\left(\frac{B_3}{B_3C_1 - B_1C_3}\right) \theta_B \frac{C_3}{B_3}$$

$$= 100 \frac{E}{L}\left(\frac{C_3}{B_3C_1 - B_1C_3}\right) \theta_B \qquad (b)$$

The effect of chord rotation may be obtained directly from the equation for moments due to sidesway, or

$$M_{AB} = \left(\frac{C_2 + C_3}{B_2C_3 - B_3C_2}\right) 1{,}000 \frac{E}{L} R \qquad (c)$$

The fixed end moment, as already derived is

$$M_{AB} = \frac{10(A_2C_3 - A_3C_2)}{B_3C_2 - B_2C_3} \qquad (d)$$

Adding equations $(a)$, $(b)$, and $(c)$ and $(d)$ the total moment $M_{AB}$ is seen to equal

$$M_{AB} = 100 \frac{E}{L}\left[\left(\frac{C_2}{B_1C_2 - B_2C_1}\right)\theta_A + \left(\frac{C_3}{B_3C_1 - B_1C_3}\right)\theta_B + \frac{10(C_2 + C_3)}{B_2C_3 - B_3C_2} R\right] + \frac{10(A_2C_3 - A_3C_2)}{B_3C_2 - B_2C_3} \qquad (e)$$

For the right or B end of the beam $M_{BA}$ may be obtained in a similar manner:

Due to $\theta_A$,

$$M_{BA} = 100 \frac{E}{L}\left(\frac{C_2}{B_1C_2 - B_2C_1}\right) \theta_A \left(\frac{B_2}{C_2}\right)$$

$$= 100 \frac{E}{L}\left(\frac{B_2}{B_1C_2 - B_2C_2}\right) \theta_A \qquad (f)$$

Due to $\theta_B$,

$$M_{BA} = 100 \frac{E}{L}\left(\frac{B_3}{B_3C_1 - B_1C_3}\right) \theta_B \qquad (g)$$

Due to chord rotation,

$$M_{BA} = \left(\frac{B_3 + B_2}{B_2C_3 - B_3C_2}\right) 1{,}000 \frac{E}{L} R \qquad (h)$$

The fixed end moment is

$$M_{BA} = \frac{10(A_2B_3 - A_3B_2)}{B_3C_2 - B_2C_3} \qquad (i)$$

Hence the total moment is

$$M_{BA} = 100\frac{E}{L}\left[\left(\frac{B_2}{B_1C_2 - B_2C_1}\right)\theta_A + \left(\frac{B_3}{B_3C_1 - B_1C_3}\right)\theta_B + \left(\frac{10(B_3 + B_2)}{B_2C_3 - B_3C_2}\right)R\right] + \frac{10(A_2B_3 - A_3B_2)}{B_3C_2 - B_2C_3} \qquad (j)$$

For a symmetrical beam, $C_3 = B_2$; $B_3 = C_2$ and $C_1 = B_1$, whence, from $(e)$,

$$M_{AB} = 100\frac{E}{L}\left[\left(\frac{C_2}{B_1C_2 - B_2B_1}\right)\theta_A + \left(\frac{B_2}{C_2B_1 - B_1B_2}\right)\theta_B + \frac{10(C_2 + B_2)}{B_2^2 - C_2^2}R\right] + \frac{10(A_2B_2 - A_3C_2)}{C_2^2 - B_2^2}$$

$$= 100\frac{E}{L}\left[\left[\frac{C_2}{B_1(C_2 - B_2)}\right]\theta_A + \left[\frac{B_2}{B_1(C_2 - B_2)}\right]\theta_B + \frac{10}{B_2 - C_2}R\right] + \frac{10(A_2B_2 - A_3\,C_2)}{C_2^2 - B_2^2} \qquad (k)$$

Similarly, for $M_{BA}$, from equation $(j)$,

$$M_{BA} = 100\frac{E}{L}\left[\left(\frac{B_2}{B_1C_2 - B_2B_1}\right)\theta_A + \frac{C_2}{C_2B_1 - B_1B_2}\theta_B + \frac{10(C_2 + B_2)}{B_2^2 - C_2^2}R\right] + \frac{10(A_2C_2 - A_3B_2)}{C_2^2 + B_2^2}$$

$$\therefore M_{BA} = 100\frac{E}{L}\left[\frac{B_2}{B_1(C_2 - B_2)}\theta_A + \frac{C_2}{B_1(C_2 - B_2)}\theta_B + \frac{10}{B_2 - C_2}R\right] + \frac{10(A_2C_2 - A_3B_2)}{C_2^2 - B_2^2} \qquad (l)$$

If the loading as well as the beam is symmetrical, $A_2$ and $A_3$ are equal. Then from equations $(k)$ and $(l)$ respectively,

$$M_{BA} = 100\frac{E}{L}\left[\frac{C_2}{B_1(C_2 - B_2)}\theta_A + \frac{B_2}{B_1(C_2 - B_2)}\theta_B +\right.$$

$$\left.\frac{10}{B_2 - C_2} R \right] - \frac{10A_2}{C_2 + B_2} \qquad (m)$$

$$M_{BA} = 100 \frac{E}{L} \left[ \frac{B_2}{B_1(C_2 - B_2)} \theta_A + \frac{C_2}{B_1(C_2 - B_2)} \theta_B + \right.$$

$$\left.\frac{10}{B_2 - C_2} R \right] + \frac{10A_2}{C_2 + B_2} \qquad (n)$$

Noting that $C_2 - B_2$ is a common factor of the first three terms of equations $(k)$, $(l)$, $(m)$ and $(n)$, these equations may be written as follows:

For a symmetrical beam, with unsymmetrical loading,

$$M_{AB} = \frac{100}{C_2 - B_2} \frac{E}{L} \left( \frac{C_2}{B_1} \theta_A + \frac{B_2}{B_1} \theta_B - 10R \right) + \frac{10(A_2B_2 - A_3C_2)}{C_2^2 - B_2^2} \qquad (o)$$

$$M_{BA} = \frac{100}{C_2 - B_2} \frac{E}{L} \left( \frac{B_2}{B_1} \theta_A + \frac{C_2}{B_1} \theta_B - 10R \right) + \frac{10(A_2C_2 - A_3B_2)}{C_2^2 - B_2^2} \qquad (p)$$

For a symmetrical beam, symmetrically loaded,

$$M_{AB} = \frac{100}{C_2 - B_2} \frac{E}{L} \left( \frac{C_2}{B_1} \theta_A + \frac{B_2}{B_1} \theta_B - 10R \right) - \frac{10A_2}{C_2 + B_2} \qquad (q)$$

$$M_{BA} = \frac{100}{C_2 - B_2} \frac{E}{L} \left( \frac{B_2}{B_1} \theta_A + \frac{C_2}{B_1} \theta_B - 10R \right) + \frac{10A}{C_2 + B_2} \qquad (r)$$

APPENDIX I: Notation:

$I$ = moment of inertia (assumed to vary along span);

$M_B$ = moment at any section due to transverse loads, considering beam as end supported;

$M_L$ = fixed end moment at left end of beam;

$M_R$ = fixed end moment at right end of beam;

$E$ = modulus of elasticity (assumed constant for entire beam);

$a_1$ = area under $\frac{M_B}{I}$ curve;

$a_2$ = moment of $a_1$ about left end of span;
$a_3$ = moment of $a_1$ about right end of span;
$b_1$ = area under $\frac{M}{I}$ curve when moment varies linearly from $M_L$ at the left end of the beam to zero at the right end of the beam;
$b_2$ = moment of $b_1$ about left end of span;
$b_3$ = moment of $b_1$ about right end of span;
$c_1$ = area under $\frac{M}{I}$ curve when moment varies linearly from $M_R$ at the right end of the beam to zero at the left end of the beam;
$c_2$ = moment of $c_1$ about left end of span;
$c_3$ = moment of $c_1$ about right end of span;

$$A_1 = \frac{1}{2}\frac{M_0}{I_0} + \frac{M_1}{I_1} + \frac{M_2}{I_2} + \frac{M_3}{I_3} + \frac{M_4}{I_4} + \frac{M_6}{I_6} + \frac{M_7}{I_7} + \frac{M_8}{I_8} + \frac{M_9}{I_9} + \frac{1}{2} + \frac{M_{10}}{I_{10}}$$ where $M_0$, $M_1$, etc., are values of $M_B$;

$$B_1 = \frac{5}{I_0} + \frac{9}{I_1} + \frac{8}{I_2} + \frac{7}{I_3} + \frac{6}{I_4} + \frac{5}{I_5} + \frac{4}{I_6} + \frac{3}{I_7} + \frac{2}{I_8} + \frac{1}{I_9};$$

$$C_1 = \frac{1}{I_1} + \frac{2}{I_2} + \frac{3}{I_3} + \frac{4}{I_4} + \frac{5}{I_5} + \frac{6}{I_6} + \frac{7}{I_7} + \frac{8}{I_8} + \frac{9}{I_9} + \frac{5}{I_{10}};$$

$$A_2 = \frac{1}{6} + \frac{M_0}{I_0} + \frac{M_1}{I_1} + \frac{2M_2}{I_2} + \frac{3M_3}{I_3} + \frac{4M_4}{I_4} + \frac{5M_5}{I_5} + \frac{6M_6}{I_6} + \frac{7M_7}{I_7} + \frac{8M_8}{I_8} + \frac{9M_9}{I_9} + 4.83\frac{M_{10}}{I_{10}}$$ where $M_0$, $M_1$, etc., are values of $M_B$;

$$B_2 = \frac{1.6}{I_0} + \frac{9}{I_1} + \frac{16}{I_2} + \frac{21}{I_3} + \frac{24}{I_4} + \frac{25}{I_5} + \frac{24}{I_6} + \frac{21}{I_7} + \frac{16}{I_8} + \frac{9}{I_9};$$

$$C_2 = \frac{1}{I_1} + \frac{4}{I_2} + \frac{9}{I_3} + \frac{16}{I_4} + \frac{25}{I_5} + \frac{36}{I_6} + \frac{49}{I_7} + \frac{64}{I_8} + \frac{81}{I_9}\frac{48.3}{I_{10}};$$

$$A_3 = 4.83\frac{M_0}{I_0} + 9\frac{M_1}{I_1} + 8\frac{M_2}{I_2} + 7\frac{M_3}{I_3} + 6\frac{M_4}{I_4} + 5\frac{M_5}{I_5}$$

$$+ 4\frac{M_6}{I_6} + 3\frac{M_7}{I_7} + 2\frac{M_8}{I_8} + \frac{M_9}{I_9} + \frac{1}{6} + \frac{M_{10}}{I_{10}}$$ where $M_0$, $M_1$, etc., are values of $M_B$;

$$B_3 = \frac{48.3}{I_0} + \frac{81}{I_1} + \frac{64}{I_2} + \frac{49}{I_3} + \frac{36}{I_4} + \frac{25}{I_5} + \frac{16}{I_6} + \frac{9}{I_7} + \frac{4}{I_8} + \frac{1}{I_9};$$

$$C_3 = \frac{9}{I_1} + \frac{16}{I_2} + \frac{21}{I_3} + \frac{24}{I_4} + \frac{25}{I_5} + \frac{24}{I_6} + \frac{21}{I_7} + \frac{16}{I_8} + \frac{9}{I_9} + \frac{1.6}{I_{10}};$$

$L$ = length of a member;

$d$ = relative movement of the ends of a member, measured normally to the original direction of a member, due to sidesway;

$R = \frac{d}{L}$ = angular rotation of chord joining ends of member, due to sidesway;

$C$ = carry over factor ($C_{LR}$ = carry over factor from left to right; $C_{RL}$ = carry over factor from right to left);

$M_{AB}$ = moment at $A$ end of beam $AB$;
$M_{BA}$ = moment at $B$ end of beam $AB$;
$\theta_A$ = slope at $A$ end of beam $AB$, referred to original position of chord $AB$;
$\theta_B$ = slope at $B$ end of beam $AB$, referred to original position of chord $AB$;
$\alpha$ = coefficient to $\theta_A$ in slope deflection equation for $M_{AB}$
$\beta$ = coefficient to $\theta_B$ in slope deflection equation for $M_{AB}$
$\gamma$ = coefficient to $R$ in slope deflection equation for $M_{AB}$
$\alpha'$ = coefficient to $\theta_A$ in slope deflection equation for $M_{BA}$
$\beta'$ = coefficient to $\theta_B$ in slope deflection equation for $M_{BA}$
$\gamma'$ = coefficient to $R$ in slope deflection equation for $M_{BA}$

# MINIMUM VELOCITIES FOR SEWERS

**Final Report of Committee to Study Limiting Velocities of Flow in Sewers**

**March 4, 1942**

## INTRODUCTORY

THIS Committee was appointed in accordance with a vote of the Sanitary Section of the Boston Society of Civil Engineers at the annual meeting on March 6, 1940. A Progress Report was submitted on March 5, 1941, at which time the Committee was continued and directed to submit a final report on or before the Annual Meeting of the Sanitary Section in 1942.

The studies and findings of the Committee have been confined for the most part to limiting *minimum velocities* of flow in *separate sewers* providing for the discharge of *normal sewage.*

It should be noted that in general the regulations of the State Health Departments and the recommendations of engineers are based on vitrified sewer pipe presumably laid with cement or compound jointing materials and with reasonably good workmanship.

In the main body of the report may be found a summary of the results of the investigations undertaken by the Committee and the conclusions reached. Following the report the material accumulated by the Committee has been assembled and abstracted according to the following captions:

| Appendix No. | Title |
|---|---|
| 1. | REGULATIONS OF STATE HEALTH DEPARTMENTS |
| 2. | BASIS OF DESIGN USED BY ENGINEERS |
| 3. | EXPERIENCE WITH FLAT SEWER GRADES |
| 4. | THEORETICAL CONSIDERATIONS |
| 5. | BIBLIOGRAPHY AND ABSTRACTS |

## REGULATIONS OF STATE HEALTH DEPARTMENTS

Inquiries were sent out by Mr. Walter Merrill to the various State

Health Departments as to their regulations and requirements for the design of sewers. Replies have been received from forty-two states and from the District of Columbia. No replies were received from the States of Idaho, Iowa, Mississippi, New Hampshire, South Dakota and Wyoming.

The replies received indicate that there are fifteen states which have adopted some form of rules and regulations concerning the design of sewers. In most of these states the regulations are in the nature of recommendations rather than hard-and-fast rules. Exceptions are provided for to meet local conditions and requirements. Of these fifteen states, eleven recommend that the design of sewers provide for a minimum velocity of 2 ft. per second with the sewer flowing full or half full. In all cases where regulations have been adopted, it is evident that a friction factor of $n = .013$, in the Kutter formula, has been used as a basis for the slopes recommended.

In addition to the fifteen states reporting the use of rules and regulations for the design of sewers, there are fifteen other states which apparently require submission of plans for approval and base their review of sewer designs on definite standards.

The minimum permissible velocity for use in the design of sewers flowing full, or half full, is 1.5 ft. per second in one state, 1.6 ft. per second in one state, and 2.0 ft. per second in twenty-one states. In one state a minimum velocity of 2.5 ft. per second, flowing full or half full, is recommended. As far as could be determined from the data submitted, a friction factor of $n = .013$ in the Kutter formula, was considered satisfactory for use in all of the states except one.

To summarize, the State Health Departments, as a general rule, either require or recommend that the design of sewers be based on slopes which will give a velocity of at least 2.0 ft. per second, flowing full or half full, assuming a friction factor of $n = .013$ in the Kutter formula. Designs based on an average velocity of as low as 1.5 ft. per second, flowing full or half full, are at times permitted under particular circumstances.

A summary of the pertinent data submitted by the various states has been tabulated in Table 4 of Appendix 1.

## Basis of Design Used by Engineers

The Committee sent out questionnaires to a number of the private engineering offices in the country, as well as to many municipal engineers, requesting an opinion as to the normally desirable and absolute minimum grades of sewers particularly in sizes from 8 to 24 inches in diameter. In a round table discussion conducted by Sewage Works Engineering at our request the specific question was asked, "What minimum slope do you believe advisable on (a) 8-inch, (b) 12-inch, (c) 15-inch sewer pipe in order to prevent objectionable deposits?"

The detailed returns from our inquiries are presented in Appendix 2. In this appendix the opinions expressed by various engineers in response to an inquiry by Metcalf & Eddy in 1913 are also reproduced.

The replies to our questionnaire are given in Table 5 and those appearing in the "Round Table" of Sewage Works Engineering are given in Table 6. For comparative purposes the minimum slopes recommended by engineers for use with 8-inch sewers have been tabulated in Table 7. It may be seen from the tables that the majority of engineers recommend a minimum slope of 4 ft. per 1000 ft. for 8-inch pipe. The corresponding velocity flowing full assuming $n = .013$ is 2 ft. per second and for $n = .015$ it is about 1.65 ft. per second.

Several of the engineers who use a flatter slope than 4 ft. per 1000 ft. for 8-inch pipe do so with the acknowledgment of possible difficulties in operation. Thus the admonition is given for an 8-inch pipe at a slope of 2.5 ft. per 1000 ft. that it is the minimum slope advisable with automatic flushing. With reference to this same slope another engineer reports that it is the absolute minimum and that some trouble is to be anticipated with the sewer at this grade. In an "extreme case" one engineer permits a slope of 2 ft. per 1000 ft. although another engineer uses this slope as the minimum without qualification.

It is interesting to note that the consulting engineers who have replied to our inquiry, practically without exception, recommend a minimum slope for an 8-inch sewer of at least 4 ft. per 1000 ft. Assuming $n = .013$ the velocity flowing full or half full at this slope would be 2 ft. per second. The minimum slopes recommended by the consulting engineers for sewers up to 24 inches in diameter are also predicated upon the above design assumptions.

## Experience with Flat Sewer Grades

A form letter of inquiry with respect to experience with flat sewer grades was sent out by the Committee to a considerable number of engineers and superintendents responsible for operation of sewerage systems. In addition at the suggestions of the Committee a Round Table discussion on this subject was conducted in Sewage Works Engineering. In the "Round Table" discussion the following questions were presented:

1. Have you experienced any trouble from deposits in sewers which you attribute to flat slopes and low velocities?
2. How often should such lines be flushed out to clear objectionable deposits?

The returns from these two lines of investigation have been abstracted in Appendix 3.

It has been found very difficult to evaluate the returns received. To a superintendent in a small community flushing the sewers as often as once a month may not be objectionable. On the other hand flushing the sewers more often than once a year in a large municipality might seem troublesome and laborious to the engineer in charge of the larger system. As expressed by Rawn, "the magnitude of a project materially influences its maintenance, and while a small organization might keep its finger closely on the pulse of a small system, a larger organization would have a great deal more difficulty keeping close contact with a much larger system. Trouble would undoubtedly be the result."

The following tabulation is a summary of the response to the survey on flushing and cleaning needs reported in detail in Appendix 3:

| | |
|---|---|
| Des Moines, Iowa | Seven 10-inch sewers laid at slopes of 1.2 to 2.0 ft. per 1000 give trouble. Six 12-inch sewers laid at slopes of 1.2 to 2.2 ft. per 1000 give trouble. |
| Tampa, Florida | Some particular sections require rodding and flushing two or three times per year. |
| Gloucester, Mass. | One 12-inch sewer at a slope of 2 ft. per 1000 requires flushing every month. Ten 8-inch sewers at slopes of 3 to 5 ft. per 1000 require flushing at least every three months. |
| Macon, Georgia | The lines are flushed twice a week to clear objectionable deposits. |

| | |
|---|---|
| Davenport, Iowa | Sewers where trouble is experienced are flushed about twice a year. |
| Fort Wayne, Indiana | Lines where deposits are troublesome are flushed every six months. |
| Ansonia, Conn. | Troublesome lines are flushed about once a month. |
| Colorado Springs, Colo. | Lines should be inspected daily and flushed each week. |
| Stockton, Cal. | About 100 stoppages a year are experienced in the main and lateral sewers. |
| Upland, Cal. | No trouble with stoppages. Lines are flushed once a year. |
| Fort Collins, Colo. | Lines are flushed every three months. |
| Huntington Park, Cal. | Street flushers are used once a month. |
| Helena, Ark. | Manual flushing every month generally. |
| Lafayette, Ind. | Troublesome lines flushed at least once a year. |
| Redendo Beach, Cal. | Necessary to flush lines with flat slopes every two weeks. |
| South Gate, Cal. | Objectionable deposits are removed from sewers by running a ball through when we flush every six to twelve months. |
| San Jose, Cal. | Certain sewers are flushed about twice during the summer period. |
| Modesto, Cal. | Flushing employed at irregular intervals as deposits occur. |
| Redlands, Cal. | Flushing about every three months appears to be necessary. |
| Lake Charles, La. | Flat lines flushed after each rain which means about twice a month. |
| Orange, Cal. | Lines dragged at least once a year. |
| Fresno, Cal. | A crew of 2 men required to keep sewerage system clean and clear. |

It is impossible to derive any broad conclusion from the data reported. However, it appears to be the consensus of opinion of those engineers and superintendents responsible for the operation and maintenance of sewerage systems that the sewers should be in general designed to give a full velocity of at least 2 ft. per second (presumably with $n = .013$).

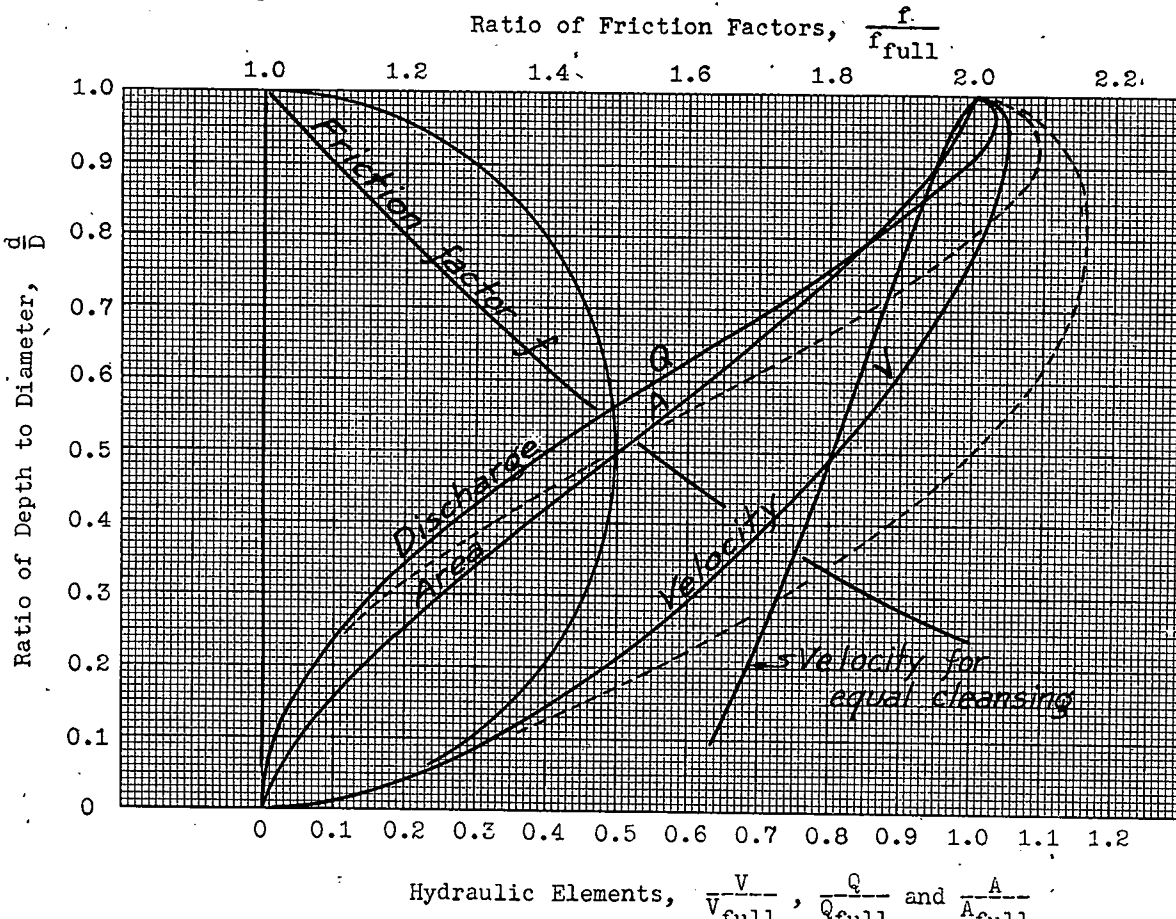
## Theoretical Considerations

A study of the modern theory of fluid turbulence has been made in order to throw light upon friction factors and cleansing or scouring velocities for sewers. The details of this study are presented in Appendix 4.

As a result of this study it appears that the only reliable experimental measurements of the friction factor available for sewers flowing full are those of Wilcox on 8-inch clay and concrete sewer pipe and Yarnell and Woodward on open-butt-joint concrete and clay drain tile 4 to 12 inches in size. These data indicate that the Weisbach-Darcy friction factor for the full pipe $f_{full}$ is independent of the material and has a value of 0.019 to 0.025 (Kutter's or Manning's $n = 0.010$ to 0.011) for 8-inch to 12-inch pipes. The corresponding magnitude of the *effective* roughness projection $k$ on the pipe walls computed from the turbulent flow theory, equation (16), Appendix 4, is about 0.4 mm. The *effective* roughness corresponding to $n = 0.013$, commonly used for the design of small sewers, is shown to be about ⅛ inch (3 mm.) and is larger than can reasonably be expected in actual sewers.

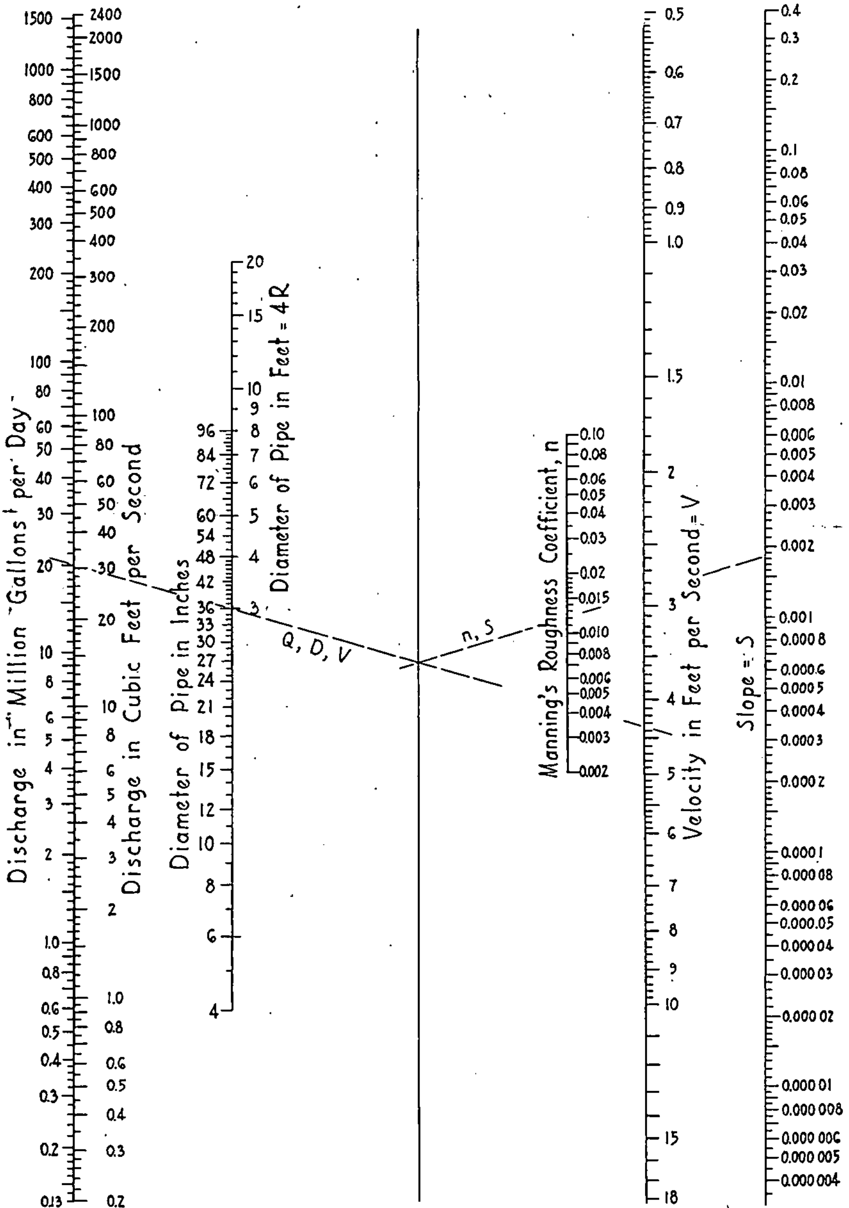
A study of the data from the experiments of Wilcox and of Yarnell and Woodward for partially full pipes shows that the friction factor $f$ increases greatly for the same pipe as the depth of flow is decreased. A similar but smaller increase obtains for $n$. The relation between the friction factor $f$ for the partially full pipe and $f_{full}$ for the full pipe as determined from these experiments is shown on Fig. 1. The *hydraulic elements* as determined from this relation are also plotted on Fig. 1. For comparison, the corresponding velocity and discharge ratios computed with a constant value of Kutter's $n$, as is customary in general practice, are shown with broken lines. The *hydraulic elements* of Fig. 1 correspond with values of Manning's $n$ which increase with partially full pipes as follows:

| $n$ for full pipe | $n$ for $\frac{d}{D} = 0.5$ | $n$ for $\frac{d}{D} = 0.25$ |
|---|---|---|
| 0.012 | 0.0138 | 0.0144 |
| 0.011 | 0.0127 | 0.0132 |



Ratio of Friction Factors, $\frac{f}{f_{full}}$
1.0 1.2 1.4 1.6 1.8 2.0 2.2
Ratio of Depth to Diameter, $\frac{d}{D}$
1.0 0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1 0
Friction factor f
Discharge
Q
Area
A
Velocity
V
Velocity for equal cleansing
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.1 1.2
Hydraulic Elements, $\frac{V}{V_{full}}$, $\frac{Q}{Q_{full}}$ and $\frac{A}{A_{full}}$


FIG. 1.




ALIGNMENT CHART FOR FLOW IN PIPES

MANNING'S FORMULA: $V = \frac{1.486}{n} R^{2/3} S^{1/2}$

FIG. 2.

These data show that the high values of $n$ usually obtained in sewage flow measurements are due principally to partially full pipes, and that such measurements are of little value unless considered in terms of the ratio of depth to diameter of pipe. To conform more closely with actual conditions, a lower value of $n$ should be used for full pipes in design than has been the practice. For the convenience of designers, the accompanying alignment chart, Fig. 2, is submitted. It is based on Manning's formula for full pipes and may be used with any value of Manning's $n$.

A study of the work of Shields on tractive force and bed-load transport shows that the velocity required to transport sediment is proportional to the square root of the size and net specific gravity of the particles and inversely proportional to the square root of the friction factor $f$, as indicated in equation (25), Appendix 4. The required velocity is independent of the size of the sewer but is affected by the depth of flow through the change in $f$. A curve of $\frac{V}{V_{\text{full}}}$ for equal cleansing based on this relation is plotted on Fig. 1.

A study of the data on scouring velocities in Appendices 1, 2 and 3 in the light of the curve of $\frac{V}{V_{\text{full}}}$ for equal cleansing indicates that a velocity of about 2 fps is required for adequate cleansing when a sewer is flowing full. The following minimum velocities are then indicated for adequate cleansing of partially full sewers:

TABLE 1

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{d_{\text{min}}}{D}$ = 1.0 | 0.5 | 0.45 | 0.40 | 0.35 | 0.30 | 0.25 | 0.20 | 0.15 | 0.10 |
| $V_{\text{min}}$, fps = 2.0 | 1.6 | 1.56 | 1.53 | 1.49 | 1.45 | 1.41 | 1.37 | 1.33 | 1.28 |

In this connection it should be noted that grit is deposited effectively in grit chambers at velocities up to about 1.0 fps.

It is shown that the transport of material in suspension depends upon scour from the bottom, that the rate of settling out is just compensated for by the rate of scour from the bottom. Hence, when the sewer velocity becomes too low, material will settle out. This material will be picked up again when the velocity is increased. Therefore if the velocity for several hours each day is in excess of that required to keep the sewer clean, the material deposited during the hours of extreme minimum flow will be picked up and carried on within a 24-hour period. If the sewer is designed to obtain a cleansing velocity at the average rate of flow throughout 24 hours on the day when the flow is a minimum, no trouble should result from deposits.

In the case of lateral sewers with only a few house connections, the average discharge throughout 24 hours may be too small to be useful for design. The discharge from a single house will be intermittent, depending upon the use of plumbing fixtures. Each day, according to studies made by Roy B. Hunter (Report BMS79, National Bureau of Standards, Nov. 5, 1941), the water demand to a house with a kitchen sink and a bathroom group with tank type water closet may be expected to exceed 7 gpm at least once. The discharge to the house drain will exceed this figure due to flush waves from the water closet. Since these flush waves are dissipated in about 100 ft. of pipe, judging by the results of experiments by E. W. Lane and O. J. Baldwin (Eng. News-Record, 116, 848, 1936), it is probably not safe to assume a discharge in excess of 7 gpm per house for sewer design.

The discharge to a sewer from several house connections will be less than the discharge per house times the number of houses, because of the improbability of synchronized discharge. If 7 gpm be accepted as the design discharge for a single house the discharge rates for several houses, based on probability studies made by Hunter, may be taken as shown in Table 2.

TABLE 2

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Number of houses | 1 | 2 | 3 | 4 | 5 | 10 | 20 |
| Discharge, gpm | 7 | 12 | 17 | 21 | 25 | 38 | 57 |
| Discharge, cfs | 0.016 | 0.027 | 0.038 | 0.047 | 0.056 | 0.085 | 0.127 |
| Min. slope, 8″ pipe, ft./1000 | 14 | 11 | 8.5 | 7.4 | 6.7 | 4.8 | 3.8 |

If sewers are designed to obtain a cleansing velocity at the minimum discharge, the full velocity and in turn the slope will be determined from the ratio of the minimum flow to the full capacity of the sewer. For each value of $\frac{d_{min}}{D}$, there are, from Fig. 1, corresponding

TABLE 3

SLOPES REQUIRED AT MINIMUM FLOW TO DEVELOP CLEANSING VELOCITIES EQUIVALENT TO 2.0 fps WITH THE PIPE FULL

Manning's $n = 0.012$ for pipe full

| $\frac{d_{min}}{D}$ | $\frac{Q_{min}}{Q_{full}}$ | $\frac{A_{min}}{A_{full}}$ | $V_{min}$ fps | $V_{full}$ fps | 6″ | | 8″ | | 10″ | | 12″ | | 15″ | | 18″ | | 21″ | | 24″ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $Q_{min}$ cfs | Slope ft./1000 | $Q_{min}$ cfs | Slope ft./1000 | $Q_{min}$ cfs | Slope ft./1000 | $Q_{min}$ cfs | Slope ft./1000 | $Q_{min}$ cfs | Slope ft./1000 | $Q_{min}$ cfs | Slope ft./1000 | $Q_{min}$ cfs | Slope ft./1000 | $Q_{min}$ cfs | Slope ft./1000 |
| 0.50 | 0.40 | 0.50 | 1.6 | 2.0 | 0.157 | 4.17 | 0.28 | 2.84 | 0.436 | 2.10 | 0.627 | 1.65 | 0.985 | 1.23 | 1.42 | 0.963 | 1.92 | 0.785 | 2.51 | 0.658 |
| 0.45 | 0.33 | 0.437 | 1.56 | 2.06 | 0.134 | 4.43 | 0.238 | 3.02 | 0.372 | 2.23 | 0.535 | 1.75 | 0.838 | 1.305 | 1.20 | 1.02 | 1.64 | 0.833 | 2.14 | 0.698 |
| 0.40 | 0.27 | 0.375 | 1.53 | 2.15 | 0.112 | 4.82 | 0.201 | 3.29 | 0.313 | 2.43 | 0.451 | 1.91 | 0.706 | 1.42 | 1.016 | 1.11 | 1.38 | 0.907 | 1.80 | 1.760 |
| 0.35 | 0.21 | 0.313 | 1.49 | 2.25 | 0.091 | 5.29 | 0.163 | 3.60 | 0.254 | 2.67 | 0.366 | 2.10 | 0.573 | 1.56 | 0.826 | 1.22 | 1.12 | 0.994 | 1.46 | 0.833 |
| 0.30 | 0.155 | 0.255 | 1.45 | 2.40 | 0.0725 | 6.01 | 0.1295 | 4.10 | 0.202 | 3.03 | 0.290 | 2.38 | 0.455 | 1.77 | 0.656 | 1.39 | 0.888 | 1.13 | 1.163 | 0.946 |
| 0.25 | 0.111 | 0.196 | 1.41 | 2.56 | 0.0542 | 6.84 | 0.0968 | 4.65 | 0.151 | 3.45 | 0.217 | 2.71 | 0.340 | 2.01 | 0.489 | 1.58 | 0.663 | 1.286 | 0.868 | 1.076 |
| 0.20 | 0.075 | 0.144 | 1.37 | 2.82 | 0.0386 | 8.30 | 0.068 | 5.65 | 0.1075 | 4.18 | 0.155 | 3.28 | 0.243 | 2.44 | 0.349 | 1.91 | 0.473 | 1.56 | 0.620 | 1.305 |
| 0.15 | 0.042 | 0.095 | 1.33 | 3.24 | 0.0248 | 10.95 | 0.0442 | 7.46 | 0.0688 | 5.52 | 0.099 | 4.33 | 0.155 | 3.22 | 0.224 | 2.53 | 0.303 | 2.06 | 0.396 | 1.72 |
| 0.10 | 0.020 | 0.053 | 1.28 | 3.90 | 0.0133 | 16.7 | 0.0238 | 11.4 | 0.037 | 8.4 | 0.0533 | 6.6 | 0.083 | 4.8 | 0.119 | 3.8 | 0.16 | 3.1 | 0.21 | 2.6 |

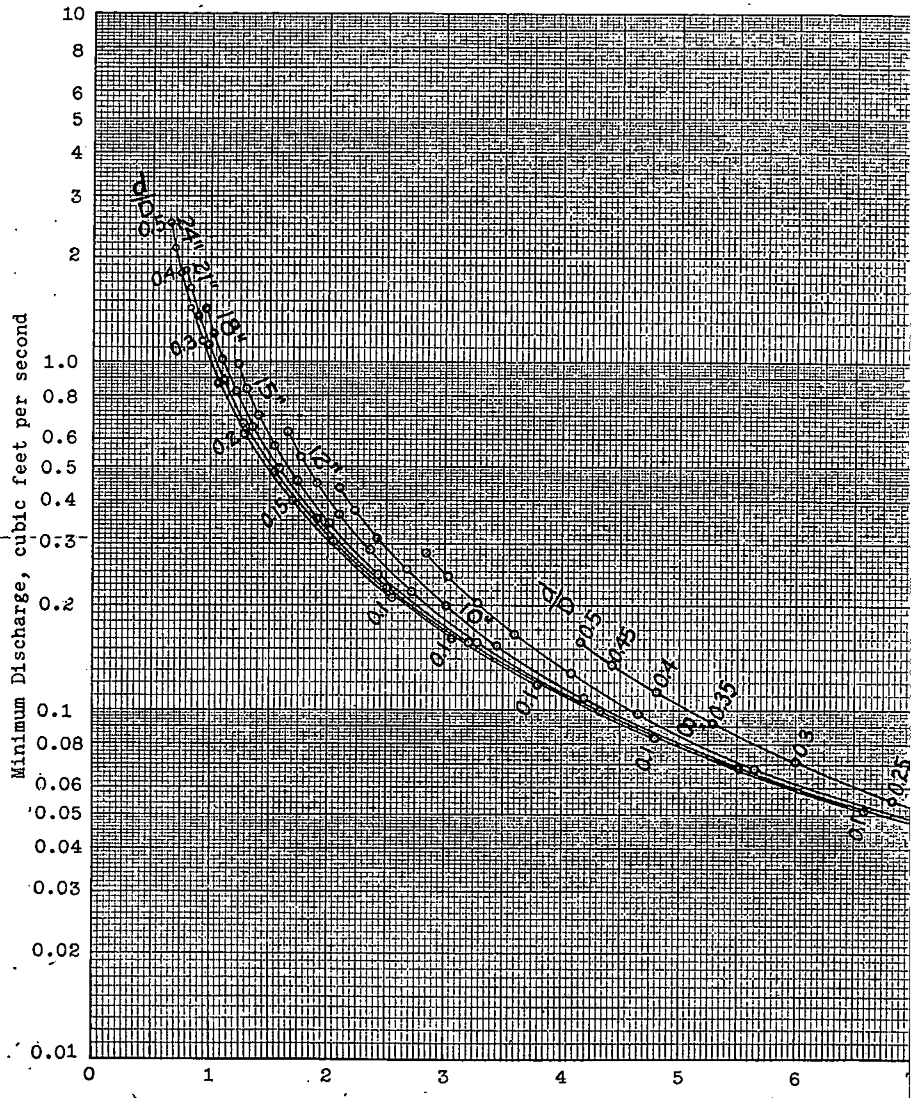
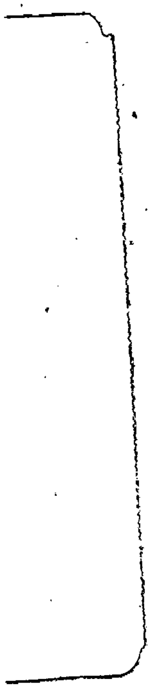
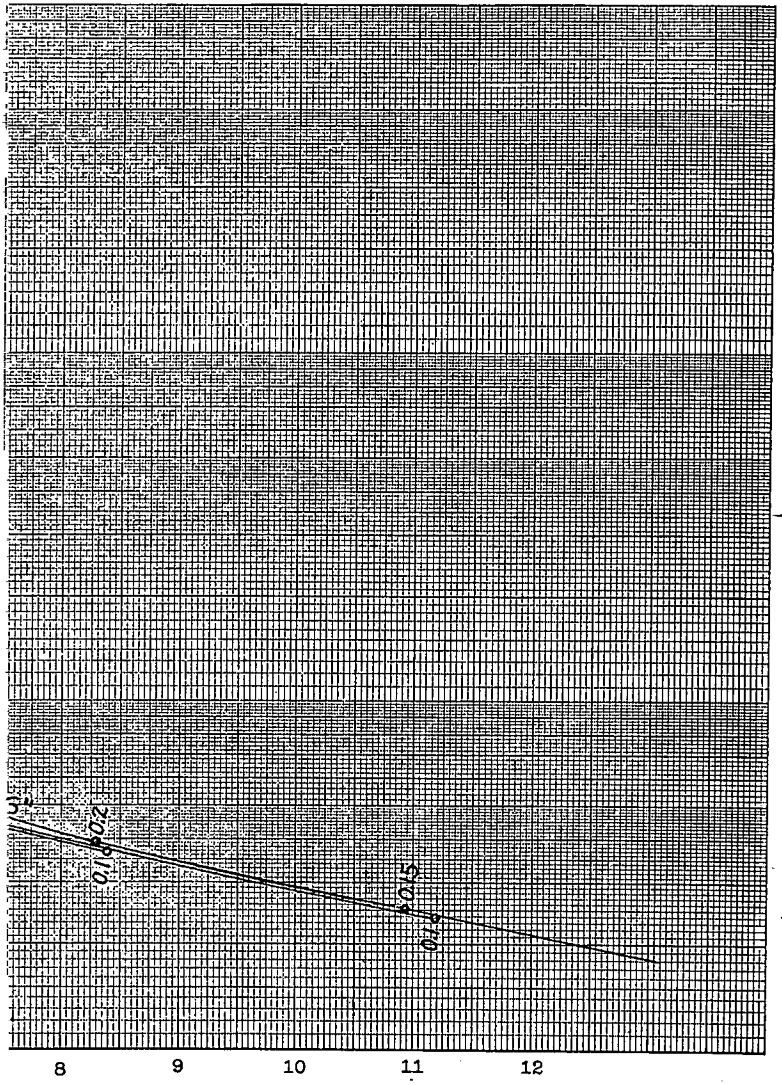



Fig. 3.—Slopes Required for Cleansing at M
in Full Pipe at 2.0 Feet
Manning's $n$=0.012

Fig 3






)0 feet

M DISCHARGE EQUAL TO CLEANSING

ECOND VELOCITY.

LL PIPE.

values for $\frac{Q_{\text{min}}}{Q_{\text{full}}}$, $\frac{A_{\text{min}}}{A_{\text{full}}}$ and $\frac{V_{\text{min}}}{V_{\text{full}}}$. Hence for each value of $V_{\text{min}}$ for adequate cleansing, there is a corresponding value for $V_{\text{full}}$ and for each pipe size there is a corresponding value for $Q_{\text{min}}$. Table 3 shows the pertinent data for partially full pipes for the values $\frac{d_{\text{min}}}{D}$ listed in Table 1. The minimum slopes in Table 3 have been computed for Manning's $n = 0.012$ for the full pipe. For convenience in interpolating between values listed in Table 3, these data have been plotted as curves on Fig. 3.

As an example of the use of Fig. 3 in design, let it be required to determine the minimum pipe size and the minimum slope for a maximum discharge of 1.10 cfs and a minimum discharge of 0.10 cfs. Solution:

With $Q_{\text{min}} = 0.10$, from Fig. 3,

For an 8″ pipe, $S = 4.7$ ft./1000 and $\frac{d_{\text{min}}}{D} = 0.26$

From Fig. 1, the corresponding value of $Q_{\text{full}} = \frac{0.10}{0.12} = 0.833$ cfs.

Hence an 8″ pipe is inadequate.

For a 10″ pipe, from Fig. 3, $S = 4.35$ ft./1000 and $\frac{d_{\text{min}}}{D} = 0.19$

From Fig. 1, the corresponding value of $Q_{\text{full}} = \frac{0.10}{0.063} = 1.58$ cfs.

The 10″ pipe is adequate, and from Fig. 1 for

$\frac{Q_{\text{max}}}{Q_{\text{full}}} = \frac{1.10}{1.58} = 0.70$, $\frac{d_{\text{max}}}{D} = 0.69$, and

$V_{\text{max}} = 0.95 \times 2.9 = 2.75$ fps.

For lateral sewers with few house connections, the minimum slopes for 8-inch pipe required for the discharge rates given in Table 2 have been estimated from Fig. 3 and are shown in Table 2.

## Summary and Conclusions

1. The regulations of the state health departments as a general rule either require or recommend that the design of sewers provide

for a minimum velocity of 2 ft. per second with the sewer flowing full or half full.

2. In all cases where state regulations have been adopted it is evident that a friction factor of $n = .013$ in the Kutter formula has been used as a basis for the slopes recommended.

3. The majority of designing engineers who have responded to our inquiries recommend that the slope of sewers provide for a minimum full velocity of 2 ft. per second and use a friction factor of $n = .013$ in the Kutter formula as a basis of design.

4. It appears to be the consensus of opinion of those engineers and superintendents responsible for the operation and maintenance of sewer systems that the sewers should be in general designed to give a full velocity of at least 2 ft. per second assuming $n = .013$ in the Kutter formula. This basis of design results in the following minimum slopes:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Size of Pipe, inches | 6 | 8 | 10 | 12 | 15 | 18 | 21 | 24 |
| Slope, ft. per 1000 ft. | 6.3 | 4 | 2.9 | 2.2 | 1.55 | 1.2 | 0.95 | 0.8 |

5. From the Theoretical Considerations, the Committee recommends that the value of $n$ for the design of full pipes be selected to conform more closely with values obtained experimentally on pipes flowing full; and that the increase in $n$ with partially full pipes be taken account of in design by the use of *hydraulic element* charts constructed for this purpose such as Fig. 1.

6. It is suggested that the following values of Manning's $n$ (Kutter's $n$ and Manning's $n$ have approximately the same value except for small pipes at flat slopes) be generally used for full pipes in the design of sewers of circular cross section, except in cases where the designer has more definite information about the value of the friction factor for the pipes and joints to be used:

| Size of Pipe | Manning's $n$ |
|---|---|
| 6-in. to 24-in. | 0.012 |
| larger sizes | 0.011 |

These values of $n$ are higher than the values indicated by the best experiments available in order to allow for conditions encountered in ordinary practice. Fig. 2 may be used for full pipes with any value of Manning's $n$.

7. For adequate cleansing, it is recommended that in general

sewers be designed with slopes at least great enough to develop a cleansing velocity at the minimum 24-hour average flow (i.e. the average discharge during 24 hours on the day when the flow is a minimum) or, in the case of laterals with few house connections, at the discharge rates indicated in Table 2.

For certain sewers the available slopes are so limited that cleansing velocities may be unobtainable with the normal sewage flow during the early years of operation. In such cases consideration should be given to the water available for flushing. The quantity of water required and the rate of flow depend on the size, slope and length of sewer to be flushed and the amount of deposit to be removed. Sewers of this type should be inspected frequently, and flushed often enough to ensure the effectiveness of the available flushing facilities.

8. It is recommended that sewers as a general rule should not be designed and constructed so as to require flushing regularly at their designed capacities. In certain cases the relative economy of permanent periodic flushing and of laying sewers at steeper slopes involving deeper excavation and possibly with provision for pumping, should be taken into consideration. However, in such cases due weight should be given to the nuisance involved in flushing and the relative unreliability of flushing as a means of maintaining the sewers in service.

9. For general practice in the design of sanitary sewers, it is recommended that the velocity at the minimum flow be sufficient to give cleansing equal to that obtained with a velocity of 2.0 fps when the pipe is flowing full. The required velocities for partially full pipes are given in Tables 1 and 3. The slopes required to develop these velocities, assuming Manning's $n = 0.012$ for pipes flowing full, are shown in Table 3 and Fig. 3.

Respectfully submitted,

FRANK L. FLOOD, *Chairman*

Committee:

WALTER E. MERRILL, *Secretary*
GEORGE F. BROUSSEAU
PROF. THOMAS R. CAMP
HERMAN G. DRESSER
RALPH W. HORNE
PAUL F. HOWARD
GEORGE A. SAMPSON

## APPENDIX 1

### Requirements of State Health Departments

The information received from the various state health departments has been summarized and tabulated as given in Table 3 of this report. In addition to the data tabulated the replies from the state health department officials included statements of interest pertaining to the problem. These replies have been abstracted and included in the appendix.

ALABAMA DEPARTMENT OF PUBLIC HEALTH, ARTHUR N. BECK, ASSISTANT SANITARY ENGINEER, BUREAU OF SANITATION

"Complying with your request of October 9 we are enclosing our 'Items Considered in the Review of Plans and Specifications on Sewer Systems and Sewage Treatment Works by Alabama State Board of Health.' "

ARIZONA STATE BOARD OF HEALTH, F. C. ROBERTS, JR., STATE SANITARY ENGINEER

"Our state has two definite temperature zones. The northern part of the state is on a high plateau with temperatures similar to those in the eastern states. The southern half of the state experiences temperatures very much on the sub-tropic. From these two factors it is at once apparent that sewage may be conveyed at a slower velocity in the lower temperatures than it can be in the higher temperatures, due to the decomposition factors. In the southern part of the state sewage that is six hours old is very old sewage, whereas in the northern part of the state that is relatively fresh. From these factors it is apparent that a treatise might be written on the subject which would prevent the establishing of any set standards."

CALIFORNIA STATE DEPT. OF PUBLIC HEALTH, C. G. GILLESPIE, CHIEF, BUREAU OF SANITARY ENGINEERING

"This Department has no rules and regulations relating to design and construction of sewers. When our opinion is asked, we recommend grades to give minimum velocities of 2 feet per second. Many of our cities and towns in the flat valleys have sewers on grades as low as

0.1 foot per hundred for 6 inch lines and correspondingly flat for other sizes."

### Connecticut Department of Health, Warren J. Scott, Director, Bureau of Sanitary Engineering

"I am sending you under separate cover one of our pamphlets on 'Sewage Disposal.' At the end of this you will find some suggested information. There are also some data on page 32 having to do with sizes and grades of sewers.

"We may make modifications of these figures to suit local conditions if necessity should arise."

### Delaware State Board of Health, R. C. Beckett, State Sanitary Engineer, Division of Sanitary Engineering

"We do not have any State Board of Health rules or regulations relating to the design and construction of sewers. In the approval of plans, we have been guided by the regulations adopted by the States of New Jersey and New York."

### Florida State Board of Health, G. F. Catlett, Chief Engineer

"This Department has not so far adopted any regulations in this regard. The terrain in Florida is so entirely flat that great difficulty is experienced in getting proper grades for sewerage systems. For this reason, it has been necessary to compromise in so many cases with what would be considered best standard practice, but have been unable to arrive at any figures that we might hold to in all cases. We have, therefore, considered each set of plans submitted to us for approval on its individual merits."

### Illinois Sanitary Water Board, C. W. Klassen, Technical Secretary

"In accordance with your October 9 request, I am sending you a bulletin, 'Items Considered in the Review of Sewerage Plans and Specifications Submitted to the State Sanitary Water Board,' which will give you our suggestions for design and construction of sewers, particularly as to limiting slopes and velocities."

INDIANA STATE BOARD OF HEALTH, B. A. POOLE, CHIEF ENGINEER, BUREAU OF SANITARY ENGINEERING

"We wish to advise you that we do not have any written rules and regulations regarding the design and construction of sewers.

"However, on checking plans we insist that the minimum gradient must provide a velocity of 2 feet per second when flowing full for sanitary sewers and 2½ feet per second for storm sewers with $n = .013$ in Kutter's formula. On manhole spacing we require a maximum separation of 300 feet on less than 18 inch and 400 feet on larger sewers. Manholes must also be constructed at upper ends and at all changes of slope and alignment."

KANSAS STATE BOARD OF HEALTH, ERNEST BOYCE, ENGINEER AND DIRECTOR

"While the Kansas State Board of Health has not adopted any rules or regulations covering the matter of minimum velocities, an effort is made to maintain a velocity of at least 1½ feet per second in sanitary sewers flowing full or half full.

"In a few instances permits have been given for sewers with very flat grades, where the alternative would be to install a pumping station. We have one or two examples of 12-inch sewers on 9/100 of 1% grade (.009 feet per hundred). These two installations referred to have given excellent service and have not caused any complaints from stoppage. One is the outfall sewer of Ellinwood, Kansas, and the other is the outfall sewer of Burlingame, Kansas. Field studies are contemplated to determine the actual velocities that are being secured in these sewers and to check the value of Kutter's $n$."

KENTUCKY STATE DEPARTMENT OF HEALTH, F. C. DUGAN, DIRECTOR, BUREAU OF SANITARY ENGINEERING

"Regarding minimum slopes and velocities in sewers, we have no definite regulations. However, in passing on plans I follow the attached table. In a few instances I have gone below the recommended minimum grades. However, I have always required that large flush tanks be constructed off the line of the sewer in order to wash out the section which is laid on a flat grade at regular intervals. So far we have had no reports of any stoppage in the sewers."

MAINE DEPARTMENT OF HEALTH AND WELFARE, ELMER W. CAMPBELL, D.P.H., DIRECTOR, DIVISION OF SANITARY ENGINEERING

"Answering your inquiry of October 9, please be advised that this department has no rules or regulations relating to the construction of sewers as such. The reason for this lack being that our department does not have legal supervisory power over the installation of sewers but merely may act in an advisory capacity when requested."

MICHIGAN DEPARTMENT OF HEALTH, JOHN M. HEPLER, DIRECTOR, BUREAU OF ENGINEERING

"The Department does not have any formal regulation concerning such matters. It is our policy to ordinarily require sewers to be laid on grades which will provide a velocity of about two feet per second. The minimum size which we usually accept for sanitary sewers is an 8-inch street sewer. A grade of 0.3 per cent will provide a velocity slightly less than two feet per second, but our experience indicates that such a grade is fairly satisfactory so that we have accepted this as about the minimum grade for a sewer of that diameter. We have a few cases in Michigan where sewers of this diameter have been laid on grades as low as 0.25 per cent and slightly less. It is very difficult to secure sufficient information to determine just what should be an absolute minimum for the reason that the quality of sewer maintenance varies widely from city to city. In the case of the City of Royal Oak which is a suburb of Detroit they have a very good city organization. The city manager, Mr. Edward Shafter, is an engineer of good training and experience and a man of good judgment and integrity. He has expressed an opinion that sewers of this diameter should not be laid in grades less than 0.3 per cent and has cited the experiences in the city of Royal Oak which tend to confirm this opinion. We also know in one or two other cases, notably at Muskegon Heights, where 8-inch sewers have been laid in grades as flat as 0.25 per cent that the resulting velocity has been so low as to materially affect the problem of sewer maintenance and tend to produce a septic sewage more difficult to treat in the sewage treatment plant. This statement was incorporated in the report by a firm of consulting engineers retained by the city several years ago to study their sewage treatment problem. In a few instances where we have found it necessary for financial reasons to accept sewers laid on a grade somewhat less than our recommended minimum, we have written letters to the community calling attention to the troubles likely to be encountered through construction of this kind and warning them that careful maintenance will be necessary in order to make sewers laid on such grades serviceable."

### Montana State Board of Health, H. B. Foote, Director, Division of Water and Sewage

"I am sending you a copy of our Regulation 100 which lists the recommended slopes for sewers of various sizes. We attempt in the building of all new sewers to obtain these grades or steeper grades, these listed being the minimum in each case.

"We have in some cases in Montana cities in which the slopes of sewers far exceed those shown on the enclosed list. Butte is one such city. Usually the sanitary sewers are laid with the natural grade of the streets or alleys and we have not attempted to place a maximum slope which would be permissible. We have had no complaint from that situation that the slope has been too great or the velocity too high. If you have any information on this point, we would appreciate having it."

### Nebraska State Department of Health, T. A. Filipi, Public Health Engineer

"We have no so-called minimum standards but endeavor to judge each project on its own merits. The engineers designing sewers usually try to adhere to the .3% slope and in no instance has this proven unsatisfactory. We have, however, one or two communities where the flows are exceptionally high; where slopes of ½ of this or .15% is satisfactory. This, of course, would never do for a sewer that was not laid true to line and where the flow was low.

"Another instance is a 10″ sewer laid at a .1% slope. This line leads from the sewer system about 3000′ to the treatment plant. Here again greater pains were taken in laying the sewer, and since the flow is high no difficulty is encountered with the deposition of solids in the barrel."

NEVADA STATE BOARD OF HEALTH, W. W. WHITE, DIRECTOR, DIVISION OF PUBLIC HEALTH ENGINEERING

"The Nevada State Department of Health have no rules, regulations or laws relating to the design and construction of sewers."

NEW MEXICO DEPARTMENT OF PUBLIC HEALTH, PAUL S. FOX, C.E., PUBLIC HEALTH ENGINEER

"Enclosed you will find a copy of our regulations governing water supplies and sewerage systems. You will note that we have not set up any specific requirements regarding the design or construction of sewers. Each set of plans is reviewed with local conditions being kept in mind at all times."

NORTH CAROLINA STATE BOARD OF HEALTH, WARREN H. BOOKER, DIRECTOR, DIVISION OF SANITARY ENGINEERING

"We have your request for information concerning limiting velocities of flow of sewers. This division has no rules or regulations adopted by the State Board of Health with regard to this matter."

OHIO DEPARTMENT OF HEALTH, F. H. WARING, CHIEF ENGINEER

"I would advise that we do not have any rules or regulations covering this matter but in our approval of plans, we have established a minimum velocity of 2 feet per second for sanitary sewers and 2.5 feet per second for combined sewers. This requires a grade of 0.40 per cent for 8-inch; 0.31 per cent for 10-inch and so on. There are instances, of course, where this minimum is deviated from but they are few and far between. In such instances we recommend an occasional flushing manhole along the line of a particularly flat sewer."

OREGON STATE BOARD OF HEALTH, CARL E. GREEN, STATE SANITARY ENGINEER

"The Oregon rules and regulations require the submission of plans for examination and approval, but do not, in themselves, specify minimum velocities and slopes required in all sewer construction.

"It has been our policy to require a minimum velocity of 2 feet per second wherever possible, but, in some areas, exceptions are carefully made because of the flatness of terrain."

PENNSYLVANIA STATE DEPARTMENT OF HEALTH, J. R. HOFFERT, CIVIL ENGINEER

"In reviewing plans of proposed sanitary sewer systems, our Bureau seeks, in so far as conditions permit, to secure a grade providing for a minimum velocity in sanitary sewers of 2 feet per second based upon a circular sewer flowing full or half full. This in the typical 8-inch lateral sewer requires a grade of 0.4 per cent for a flow of half depth and corresponds to a velocity of 1.33 feet per second when flowing at a depth of one-quarter diameter.

"In general, we do not have difficulty in securing such velocities but under certain conditions of topography, it has been found advisable to approve grades somewhat under the foregoing inasmuch as to secure the desired minimum grade would involve either pumping or the construction of flush tanks of somewhat doubtful operating efficacy. Where grades below the desired minimum are approved for limited areas, the permit usually calls the permittee's attention to the need for additional maintenance."

TENNESSEE STATE DEPARTMENT OF PUBLIC HEALTH, W. C. WILLIAMS, COMMISSIONER OF PUBLIC HEALTH

"For a number of years we have asked for a minimum velocity of flow of 2 feet per second. Slopes which are used for determining these velocities are in accordance with charts prepared on the basis of $n = 0.013$ as applied in Kutter's Formula."

UTAH STATE BOARD OF HEALTH, LYNN M. THATCHER, DIRECTOR, DIVISION OF PUBLIC HEALTH ENGINEERING AND SANITATION

"Regarding regulations of this department pertaining to minimum slopes and velocities in sewers we have been guided in the past by accepted hydraulic standards such as suggested by Metcalf & Eddy, and have never bothered to prepare regulations on the subject.

"However, in many instances the commonly accepted values for minimum slopes do not meet a degree of precision ordinarily desirable in sanitary engineering work, but our limited observations would not enable statement of our conclusions."

VERMONT DEPARTMENT OF PUBLIC HEALTH, EDWARD L. TRACY, SANITARY ENGINEER

"This department has no regulation governing the velocity of flow in sewers. In fact, the laws governing the construction of sewers in this state are quite inadequate and in need of revision."

VIRGINIA DEPARTMENT OF HEALTH, RICHARD MESSER, DIRECTOR, SANITARY ENGINEERING

"This State has not adopted any rules and regulations relating to limiting velocities of flow in sewers."

WASHINGTON DEPARTMENT OF HEALTH, ROY M. HARRIS, CHIEF, DIVISION OF PUBLIC HEALTH ENGINEERING

"I regret to advise that we do not have any rules or regulations covering these requirements, and I am therefore unable to furnish you with the material requested."

WEST VIRGINIA DEPARTMENT OF HEALTH, J. B. HARRINGTON, DIRECTOR, DIVISION OF SANITARY ENGINEERING

"This department does not have any rules or regulations relative to the design or construction of sewers other than that the State law requires all sewers to be approved by the Division of Sanitary Engineering before construction is begun.

"We follow closely the design standards given in 'American Sewerage Practice' by Metcalf and Eddy."

DISTRICT OF COLUMBIA HEALTH DEPARTMENT, C. F. BROWNING, PUBLIC HEALTH ENGINEER

"I have to advise that the District of Columbia does not have any rules and regulations, such as you outline, covering the design and construction of sewers."

TABLE 4

RULES AND REGULATIONS OF STATE HEALTH DEPARTMENTS RELATING TO DESIGN OF SEWERS

| State | Regulations | Min. Size (Inches) | Value of "n" in Kutter formula | Min. Vel. ft. per second | Suggested Minimum Slopes in feet per 1000 feet Pipe Diameters | | | | | Remarks |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | 6" | 8" | 10" | 12" | 15" | |
| Alabama | None | 8 | .013 | 2.0 | | 4.0 | 2.8 | 2.2 | 1.5 | Permits for sanitary surveys required by State law. |
| Arizona | None | | | | | | | | | |
| Arkansas | Yes | | .013 | 2.0 | 6.0 | 4.0 | 2.9 | 2.2 | | |
| California | None | | | 2.0 | | | | | | |
| Colorado | Yes | 6 | .013 | | 5.0 | 4.0 | 2.8 | 2.0 | 1.5 | |
| Connecticut | None | 8 | .013 | | 6.4 | 4.0 | 2.9 | 2.2 | 1.6 | Justifiable modifications permitted. |
| Delaware | None | | | | | | | | | In approving plans, guided by New Jersey and New York regulations. |
| Florida | None | | | | | | | | | |
| Georgia | Yes | | | 2.5 | | | | | | No other data received. |
| Idaho | ? | | | | | | | | | |
| Illinois | None | 8 | .013 | 2.0 | | 3.6 | 2.7 | 2.1 | 1.5 | |
| Indiana | None | | .013 | 2.0 | | | | | | |
| Iowa | ? | | | | | | | | | |
| Kansas | None | | | 1.5 | | | | | | |
| Kentucky | None | | .013 | 1.6 | | 2.5 | 2.0 | 1.7 | 1.4 | |
| Louisiana | Yes | | .013 | 2.0 | | 4.0 | 2.9 | 2.2 | 1.6 | In flat part of southern Louisiana flatter grades permitted. |
| Maine | None | | | | | | | | | State acts only in advisory capacity when requested. |
| Maryland | Yes | 8 | .013 | 2.0 | | 4.0 | 2.9 | 2.2 | 1.6 | Justifiable modifications permitted. |
| Massachusetts | None | 8 | .015 | 2.0 | | | | | | Justifiable modifications permitted. |
| Michigan | None | 8 | | 2.0(—) | | 3.0 | | | | |
| Minnesota | Yes | 8 | .013 | 2.0 | | 4.0 | 3.0 | 2.2 | | Minimum desirable. |
| | | | | | | 2.5 | 2.3 | 2.0 | 1.6 | Minimum permissible. |
| Mississippi | ? | | | | | | | | | |
| Missouri | Yes | | | 2.0 | | | | | | Justifiable modifications permitted. |
| Montana | Yes | | .013 | 2.0 | | 4.0 | 2.9 | 2.2 | 1.6 | |
| Nebraska | None | | | | | 3.0 | | | | |

TABLE 4 (*Continued*)

| State | Regu-lations | Min. Size (Inches) | Value of "n" in Kutter formula | Min. Vel. ft. per second | Suggested Minimum Slopes in feet per 1000 feet Pipe Diameters 6" | 8" | 10" | 12" | 15" | Remarks |
|---|---|---|---|---|---|---|---|---|---|---|
| Nevada | None | | | | | | | | | |
| New Hampshire | ? | | | | | | | | | |
| New Jersey | Yes | 8 | .013 | 2.0 | | 4.0 | 2.9 | 2.2 | 1.6 | Justifiable modifications permitted. |
| New Mexico | None | | | | | | | | | |
| New York | Yes | 8 | .013 | 2.0 | | 4.0 | 2.8 | 2.2 | 1.5 | Justifiable modifications permitted. |
| North Carolina | None | | | | | | | | | |
| North Dakota | Yes | 6 | .013 | 2.0 | | 4.0 | 3.0 | 2.2 | | Minimum desirable. |
| | | | | | | 2.5 | 2.3 | 2.0 | 1.6 | Minimum permissible. |
| Ohio | None | | | 2.0 | | 4.0 | 3.1 | | | |
| Oklahoma | Yes | 8 | .013 | 2.0 | | 4.0 | 2.9 | 2.2 | 1.6 | Justifiable modifications permitted. |
| Oregon | None | | | 2.0 | | | | | | Justifiable modifications permitted. |
| Pennsylvania | None | | .013 | 2.0 | | 4.0 | | | | |
| Rhode Island | None | | | | | | | | | |
| South Carolina | Yes | | | | | | | | | Written approval of plans by State Board of Health required. |
| South Dakota | ? | | | | | | | | | |
| Tennessee | None | | .013 | 2.0 | | | | | | |
| Texas | Yes | 6 | .013 | 2.0 | 6.0 | 4.0 | 2.9 | 2.2 | 1.6 | |
| Utah | None | | | | | | | | | Guided by accepted hydraulic standards. |
| Vermont | None | | | | | | | | | Laws governing sewer construction quite inadequate. |
| Virginia | None | | | | | | | | | |
| Washington | None | | | | | | | | | |
| West Virginia | None | | | | | | | | | Follow design standards in "American Sewerage Practice", M. & E. |
| Wisconsin | Yes | 6 | | | | 3.0 | 2.2 | 1.8 | 1.4 | Follow design standards in "American Sewerage Practice", M. & E. |
| Wyoming | ? | | | | | | | | | |
| Dist. of Columbia | None | | | | | | | | | |

## APPENDIX 2

### Basis of Design Used by Engineers

The following opinions by practicing engineers have been given to the Committee in answer to our form letter of inquiry:

WHITMAN, REQUARDT & SMITH, BALTIMORE, MARYLAND

This office endeavors to hold a minimum velocity of 2 feet per second when flowing full, using $n = .013$ in Kutter's formula for vitrified clay pipe sewers. In flat country it sometimes becomes necessary to relax these standards and go as low as 1.5 feet per second in which case studies are made as to relative pumping costs, trench depths and the economics of the problem.

CURRIE ENGINEERING COMPANY, SAN BERNARDINO, CALIFORNIA

"Our experience in minimum slopes of sewers is leading us to design on much flatter grades than we thought was possible some years ago. Like most engineers, we have used very flat grades, if their use would eliminate pumping, and have had little or no trouble with them. For the most part, we do not like to go below two-tenths on eight-inch pipe. However, we have laid eight-inch on a one-tenth per cent where it was not at the extreme upper end of the system. I believe the depth of flow in the lines is fully as important as the velocity, and do not believe that velocities of one foot a second will cause any trouble in the system.

"It might also be of interest to you to know that for the past eight years we have used very few, if any, flush tanks on the upper end of the sewer system. This is considerably in contrast with the old days when we used to use them at the upper end of practically every sewer line.

"We have instructed the cities to watch the sewers and flush them with fire hose if necessary, and, eventually, to install the flush tanks if it was seen that they were needed."

BEN S. MORROW, CITY ENGINEER, PORTLAND, OREGON

"We do not lay eight-inch sewers on grades flatter than 1 per cent except in rare cases where it seems impossible to get a 1 per cent gradient, or steeper, and endeavor in all sewers to have a minimum velocity of 3 feet per second at half capacity."

RAYMOND R. RIBAL, PRINCIPAL ASSISTANT ENGINEER, EAST BAY CITIES SEWAGE DISPOSAL SURVEY, BERKELEY, CALIFORNIA

"It has been usual practice in Oakland, California, although not a fast rule, to maintain a minimum grade of .7% for sanitary sewers. In Stockton, California, the rule is somewhat more rigid and the minimum grade a little less, or .5%. Although rules like these exist for a number of western cities, necessity has frequently determined sewer grades, especially where sewage is discharged by gravity into tidal waters, as in the San Francisco Bay. Instead of slope, perhaps a better basis for design is a minimum velocity which western experience seems to indicate is between 2.0 and 2.5 feet per second."

HARRY GOODRIDGE, CITY ENGINEER, BERKELEY, CALIFORNIA

"The minimum slope at which we have laid small sewers, such as 6" and 8", is 3/10 of 1%. We do not undertake the flushing of the sewers and they do not appear to need flushing. The greatest trouble in certain areas is caused by tree roots. We have about 400 miles of sewers and three men and one truck to take care of the necessary maintenance. In laying sewers on grades as low as 3/10 of 1%, extreme care must be exercised in grading the trench and laying pipe and cementing the joints. Under no circumstances should any cement be allowed to project into the sewer at the joints. It would be desirable, wherever sewers are laid on such flat grades, that manholes be constructed at a distance not greater than 300'."

CLYDE C. KENNEDY, SAN FRANCISCO, CALIFORNIA

"While we think it advisable to secure velocities of 2.5 feet per second where the slope permits it, there are many parts of the interior valley where it is not possible to secure these velocities. Gradients of .1 foot per 100 feet are common. If purely sanitary sewage is handled this is quite satisfactory."

LLOYD ALDRICH, CITY ENGINEER, LOS ANGELES, CALIFORNIA

"For sewers in this City, constructed on minimum grades, or steeper, adequate velocities and normal maintenance are obtained when flushing structures are installed at the end of the lines. Vitrified clay pipe sewers are now installed as a general rule."

LOUIS A. GEUPEL, SUPERINTENDENT, WATER WORKS DEPARTMENT, EVANSVILLE, INDIANA

"For design and general practice we have always required the minimum grades for 8" to 36" sewers which would give the 3 feet per second flow with absolute minimum without flush tanks of 2.5 feet per second flow. In cases where 2 feet per second flow had to be used in flat ground flush tanks were required."

H. W. JORGENSEN, CITY ENGINEER, SAN DIEGO, CALIFORNIA

"For a great many years we have attempted to maintain a minimum velocity of two feet per second, and to this end have designed our sewers on grades calculated to produce such velocities when flowing half full. This means, of course, that if the flow is less than half the depth, the velocity will be materially reduced. For example, at a depth of one-quarter of the diameter the velocity will be about 1.3 feet per second.

"We have experienced very little trouble attributed to this factor alone, but feel that a minimum velocity of one and a half feet per second should be maintained even under conditions of minimum flow.

"We have had some trouble with deposits in trunk sewers on minimum grades where hillside laterals on steep grades discharged into them."

JOHN F. SKINNER, CONSULTING ENGINEER, GLENDALE, CALIFORNIA

"Most all small diameter sewers, at their upper ends, should be provided with flush-tanks. I have made a practice of installing a flush-tank at the head of any sanitary sewer in which the normal flow has a velocity less than five (5) feet per second, at the upper end. This can properly be omitted where the upper end is quite steep; say, in excess of 2½%."

PROF. H. E. BABBITT, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS

"It has been my practice to use such a slope as will give a velocity of 2 ft. per second, or more, when the sewer is full."

H. H. CORSON, CITY ENGINEER, BIRMINGHAM, MICHIGAN

"My own observation has been that velocities of less than 2 feet per second are likely to cause trouble."

WILLIAM A. HANSELL, ENGINEER OF SEWERS, ATLANTA, GEORGIA

"The terrain in Atlanta is such that we have no difficulty in obtaining grades to give us self-cleansing velocities.

"We regard two and one-half feet per second as the proper velocity to avoid stoppage, and, of course, if we had to have a lower velocity in some particular case, we would have to put it in, but our aim is minimum two and one-half feet per second."

GREELEY & HANSEN, ENGINEERS, CHICAGO, ILLINOIS

"It has been the practice for a number of years in this office to use minimum grades which, according to the Kutter formula with $n = 0.013$, give a velocity of about 2.0 feet per second with the pipe flowing full or half full."

A. S. JONES, DESIGN ENGINEER, DES MOINES, IOWA

"It has always been the policy of the engineers in charge of sewer design in Des Moines to design for a minimum velocity of two feet per second, using $n = .013$ in the Kutter formula."

CONSOER, TOWNSEND & QUINLAN, CONSULTING ENGINEERS, CHICAGO, ILLINOIS

"For combined sewers in dry sandy areas we usually try to use a little higher minimum because of the tendency of the sand to settle out especially after storms. In general our practice in dry sand is to try to get at least 25% more grade than is set out above as standard.

"In clay soils we sometimes reduce the standard minimums by 25% when necessary to avoid pumping. In this case we place special emphasis upon accurate laying of the pipes and smoothing of the interior portion of the joints.

"We have noticed that we have less trouble with deposits at minimum grades in localities where ground water is fairly high and infiltration provides a constant flow to keep the pipe bottoms wet. Under these conditions minimum grades may also be safely reduced by 25% below the standard.

"About two hundred miles of sewers in the sizes mentioned have been constructed under our supervision in the Chicago area. Practically all of these sewers have required no maintenance over a fifteen-year period."

HAVENS AND EMERSON, CONSULTING SANITARY ENGINEERS, CLEVELAND, OHIO

"Our office uses a minimum velocity in sanitary sewers of 1.75 feet per second when the sewer is flowing full or half full. This is upon the basis of a coefficient '*n*' of .015 for vitrified pipe."

LYLE PAYTON, CITY ENGINEER, STOCKTON, CALIFORNIA

"A minimum velocity of two feet per second is recommended."

HOWARD CARTER, CITY ENGINEER, SANTA MONICA, CALIFORNIA

"The minimum slope desirable depends on the flow in the sewer. Each situation must be studied for amount, character, etc. of sewage in the district."

W. W. MATHEWS, SUPERINTENDENT, GARY SANITARY DISTRICT, GARY, INDIANA

"In general all lateral sewers should be laid at such a grade that the minimum velocity should be not less than 2 feet per second and 2½ feet is preferable where natural ground slopes permit being laid on a grade that will give this velocity."

W. L. POPP, CITY ENGINEER, SAN JOSE, CALIFORNIA

"We use a minimum velocity of two feet per second (approximately a slope of 4 ft. per thousand on 8-inch lines) based on pipes flowing half full or full."

JOHN L. MASON, ASSISTANT CITY ENGINEER, MODESTO, CALIFORNIA

"Six-inch pipe is used on a slope of four feet per thousand feet for collection laterals to serve distances up to 1200 ft. long.

STANLEY SHUPE, CITY ENGINEER, KITCHENER, ONTARIO

"Our experience in minimum sewer grades is that two feet per second velocity is the minimum from which to escape periodic flushing problems."

R. H. Cason, City Engineer, Tampa, Florida

"It is my belief that the minimum slopes advisable on an 8-inch should be 0.25%; on 12-inch, 0.18%; on 15-inch, 0.15%. However, some sections of Tampa have sewers laid on much flatter grades than those stated, and such sewers have functioned entirely satisfactorily. As an actual fact, this city has much 8-inch laid on 0.20%; 12-inch on 0.12%; and 15-inch on 0.10%."

Opinion Given to Metcalf & Eddy in 1913

Extract from "American Sewerage Practice," Vol. 1, Design of Sewers, Second Edition, 1928, by Metcalf & Eddy:

James N. Hazlehurst stated that his practice had been largely in connection with sewer systems in the southeastern coast states, where there is much silt and running sand. Minimum grades were absolutely necessary to accomplish anything and he generally used grades lower than those recommended in textbooks. The minimum grade for each size of pipe sewer, which he ordinarily permitted, was: 6-in. sewer, 0.33 per cent; 8-in., 0.25; 10-in., 0.20; 12-in., 0.17; 15-in., 0.15; 18-in., 0.12; 20-in., 0.10; 24-in., 0.08. When sewers were properly constructed he reported that he knew of no trouble from deposits when the grades were not lower than those stated. In Waycross, Georgia, there were 8-in. pipe sewers on grades as flat as 0.24 per cent, which operated without giving trouble; on a few grades which were as flat as 0.10 per cent, however, the sewers were clogged from time to time and had to be rodded out.

Charles B. Burdick stated that it was the practice of Alvord and Burdick to secure grades that would give a velocity of 2 ft. per second in separate sewers flowing full or half full, and to reduce this to 1½ ft. per second if necessary. Even on such grades they used flush tanks at the summits of the laterals, and if these velocities could not be obtained, special flush tanks were usually installed. On combined sewers they endeavored to secure 3 ft. velocity, but reduced it to 2 ft. if necessary. He stated:

"It is our practice to get all the grade we can at reasonable expense, and if it is impossible through physical conditions or cost to get the grade desired, we usually install some means for flushing, with the idea of removing deposits. We have in several cases installed a specially capacious flush tank at the head of a main where an unusually flat grade is used, these especially flat grades coming more commonly on mains than laterals."

George G. Earl stated that the standard minimum grades for sewers, in New Orleans, given in Table 29, were occasionally disregarded, because it had been necessary in some cases to lay considerable 8-in. pipe on grades as low as 0.25 per cent. The aim is to have a velocity of 2 ft. per second in a half-full 8-in. pipe, and a slightly increasing velocity in half-full sewers as the size increases. The sewers were of vitrified pipe up to 30-in. diameter, and either brick or concrete in larger sizes. Some of those over 30-in. in size are semi-elliptical in section, but on account of constant infiltration the volume of flow is sufficient at all times to render circular sections fairly satisfactory.

Better bottom grades are usually obtained in the drainage system at New Orleans than in the sewers. The main drains have a V-shaped bottom, with transverse slopes of about 1:4; they are 4 to 25 ft. wide, with good bottom gradients which give velocities of 5 to 10 ft. per second when running full. The laterals enter them with inverts flush with the bottom at the sidewalls, and thus have the maximum grade practicable. Earl stated that the drainage system, particularly the vitrified pipe laterals from 10 to 30 in. in diameter, receive street

Table 29

Minimum Grades on New Orleans Sewers

| Diameter inches | Slope per cent | Diameter inches | Slope per cent | Diameter inches | Slope per cent |
|---|---|---|---|---|---|
| 8 | 0.33 | 27 | 0.100 | 48 | 0.062 |
| 10 | 0.25 | 30 | 0.091 | 51 | 0.059 |
| 12 | 0.21 | 33 | 0.083 | 54 | 0.056 |
| 15 | 0.167 | 36 | 0.083 | 57 | 0.053 |
| 18 | 0.133 | 39 | 0.077 | 60 | 0.050 |
| 21 | 0.114 | 42 | 0.071 | 63 | 0.050 |
| 24 | 0.100 | 45 | 0.067 | 66 | 0.050 |

washings and sweepings in dry weather when the flow is inadequate to remove them, and, consequently, a good deal of flushing and cleaning is required of these dry-weather accumulations.

George W. Fuller stated that his drafting-room practice for separate pipe sewers was based on a 2-ft. velocity when half-full, with a coefficient of roughness, *n*, of 0.013. This coefficient is also used for concrete sewers 24 in. in diameter and over, and 0.015 is used for brick sewers. Rather than go to the expense of pumping where the grades tend to make it necessary, the slopes giving the velocities mentioned are sometimes flattened. This is done, however, only after a careful examination of local conditions on the ground, and is not normal office practice. For instance, at Vincennes, Indiana, in a sewerage system designed about 1910, Fuller made use of grades of 0.3 per cent with 8-in. pipe and in some cases a grade of only 0.25 per cent was used. J. R. McClintock reported subsequently for Fuller that an examination of the Englewood, New Jersey, sewerage system revealed a number of sewers with very low grades, which were apparently quite satisfactory. Six-inch sewers were discharging freely with grades as low as 0.35 per cent, and there were cases of 12-in. pipe with a grade of about 0.10 per cent, and 8-in. pipe with grades of 0.10, 0.15, and 0.20 per cent were in satisfactory operation. There were other sections of this same system, however, where sewers with grades apparently no lower were partly clogged.

Fuller stated that in the case of separate sewers he was of the opinion that the depositing velocities would not have appreciable significance if substantially every day there were periods when the velocity approached 28 in. per second or more, and that he stated carefully to clients that where the slopes of sewers, more than two or three blocks removed from flush tanks at the head of a line, showed a velocity of less than 20 in. per second, care should be taken to flush the sewers either by a hose or some equivalent. In the case of combined sewers, he endeavored to secure a nominal minimum velocity of 2½ ft. per second. In practically every case where he has had occasion to study in detail the conditions of such sewers, a heavy grit has been found deposited in them. If these deposits were not removed, they apparently decomposed and became more or less cemented by ferrous sulphide. The result was that scouring velocity applicable to ordinary street wash would no longer suffice. This he found quite marked in Elizabeth, New Jersey, although the data are too meager to find place in a record of accurate information.

C. E. Grunsky stated that the minimum grades in California cities, reported to him by the engineers of the places named, were as given in Table 30. The city engineer of Stockton said that the grades

TABLE 30

MINIMUM GRADES IN CALIFORNIA CITIES, PER CENT

| Size inches | Stockton | Fresno | Modesto | Visalia | Sacramento |
|---|---|---|---|---|---|
| 6 | 0.2 | 0.15 | 0.16 | 0.3 | 0.25 |
| 8 | 0.143 | 0.1 | 0.16 | 0.24 | 0.2 |
| 10 | 0.139 | 0.1 | 0.2 | 0.143 | 0.16 |
| 12 | 0.1 | 0.1 | 0.152 | 0.143 | 0.12 |
| 15 | — | — | 0.09 | | |
| 18 | — | — | — | 0.1 | |

in that city have caused no trouble during the 25 years the sewers have been in service; these sewers carry only sewage, rain water being excluded. Once in a great while they have had some trouble from deposits at Fresno, due to sluggish flow, according to the city engineer. The city engineer of Visalia stated that he had made float measurements in the sewers and found that the actual minimum velocity when they are running one-third to one-half full, was 1.1 ft. per second in 10-in. sewers, and a velocity of 1.75 ft. per second was observed in an 18-in. sewer half full. The light grades caused no trouble in that city. The city engineer of Sacramento stated that the depth of flow in the sewers of his city did not average one-fourth of their diameters; in no case had there been any offensive deposits.

T. Chalkley Hatton in experiments with two 24-in. sewers discharging creek water carrying considerable clay, the grade being 0.077 per cent, found no appreciable sediment with the following depths in inches and velocities in feet per second:

| | | | |
|---|---|---|---|
| Depth | 5 | 12 | 12 |
| Velocity | 1.21 | 2.35 | 1.70 |

## TABLE 5
## BASIS OF DESIGN USED BY ENGINEERS

| Engineer or Authority | Diameter of Sewer in Inches and Slope in Feet per 1000 Feet 8″ | 10″ | 12″ | 15″ | 18″ | 20″ | 24″ | Remarks |
|---|---|---|---|---|---|---|---|---|
| H. H. Corson, City Engr. | 6.2 | 4.4 | 3.3 | 2.4 | 1.8 | 1.5 | 1.2 | Normal minimum. |
| Birmingham, Mich. | 4.0 | 2.8 | 2.2 | 1.5 | 1.3 | 0.9 | 0.8 | Absolute minimum. |
| Louis A. Geupel, Supt., | 6.0 | 4.2 | 3.2 | 2.4 | 1.6 | | 1.2 | Velocity 3 ft. per sec. |
| Water Works Dept., | 4.0 | 3.0 | 2.4 | 1.8 | 1.4 | | 0.9 | Velocity 2.5 ft. per sec. |
| Evansville, Ind. | 2.6 | 2.0 | 1.5 | 1.2 | 0.9 | | 0.6 | Velocity 2.0 ft. per sec. |
| John F. Skinner, Cons. Eng., | 5.0 | | 3.0 | 2.4 | 2.0 | | 1.5 | Good practice. |
| Glendale, Calif. | 4.0 | | 2.5 | 2.0 | 1.7 | | 1.2 | Absolute minimum. |
| H. W. Jorgensen, City Engr., | 4.0 | 3.0 | 2.5 | 1.5 | 1.2 | 0.85 | 0.75 | Normal desirable. |
| Glendale, Calif. | 2.5 | 1.6 | 1.2 | 0.9 | 0.7 | 0.55 | 0.45 | Absolute minimum "some trouble experienced with sewers on these grades." |
| Whitman, Requardt & Smith Baltimore, Md. | 4.0 | 2.8 | 2.2 | 1.6 | 1.2 | 1.0 | 0.8 | Standard minimum. |
| Greeley and Hansen Chicago, Ill. | 4.0 | 3.0 | 2.2 | 1.6 | 1.2 | 1.0 | 0.9 | Minimum. |
| Consoer, Townsend & Quinlan Chicago, Ill. | 4.0 | 3.3 | 2.5 | 1.6 | 1.2 | 1.0 | 0.9 | Minimum. |
| Lloyd Aldrich, City Engr., Los Angeles, Calif. | 4.0 | 3.2 | 2.4 | 1.6 | 1.2 | 1.0 | 0.8 | Minimum. |
| John W. Budd, Supt. Streets, Des Moines, Iowa | 4.0 | 2.8 | 2.2 | 1.5 | 1.2 | 1.0 | 0.8 | Minimum. |
| Milwaukee Sewerage Comm. | 4.0 | 2.8 | 2.1 | 1.5 | 1.2 | 0.9 | 0.8 | Minimum. |
| Metcalf & Eddy | 4.0 | 2.9 | 2.2 | 1.5 | 1.2 | 1.0 | 0.8 | Minimum "found safe, though steeper grades are always desirable." Sewerage and Sewage Disposal, 2d Ed., 1930, p. 140. |
| Havens & Emerson | 4.5 | 3.1 | 2.4 | 1.7 | 1.3 | 1.1 | 0.85 | Minimum velocity based on $V$=1.75 ft./sec., $n$=0.015. |
| New Orleans, La. | 3.3 | 2.5 | 2.1 | 1.67 | 1.33 | 1.14 | 1.00 | Standard minimum grades. |

TABLE 6

BASIS OF DESIGN USED BY ENGINEERS

(Abstracted from Contribution to the Round Table in Sewage Works Engineering, March and Sept., 1941)

| Engineer or Authority | Diam. of Sewer in Inches and Slope in Feet per Foot 8" | 12" | 15" | Remarks |
|---|---|---|---|---|
| R. H. Cason, City Engr., Tampa, Florida | 2.5 | 1.8 | 1.5 | Minimum advisable. |
| O. J. Semmes, Jr., City Engr., | 4.0 | 2.5 | 1.5 | Slope to use if possible. |
| Pensacola, Florida | 2.0 | 1.5 | 1.0 | Slope to use in extreme case. |
| C. R. McAnlis, City Engr., Fort Wayne, Ind. | 5.0 | 3.2 | 1.5 | Minimum recommended. |
| C. M. Jones, Supt. Sewers, Colorado Springs, Colo. | 10.0 | 5.0 | 4.0 | Minimum recommended. |
| Watson Walker, City Engr., Macon, Ga. | 4.0 | 2.0 | 1.6 | Minimum advisable. |
| V. B. Clark, City Engr., Ansonia, Conn. | 4.0 | 2.5 | 1.5 | Minimum satisfactory. |
| B. L. Bundy, City Engr., Rendendo Beach, Calif. | 4.0 | 2.4 | 1.6 | Minimum desirable. |
| A. N. Beck, Asst. San. Engr., Montgomery, Ala. | 4.0 | 2.2 | 1.5 | Minimum recommended. |
| F. E. Alderman, City Engr., South Gate, Calif. | 4.0 | 2.2 | 1.5 | Minimum where feasible. |
| W. O. Costello, Asst. Engr., Sacramento, Calif. | 3.5 to 4.0 | 2.0 to 2.5 | 2.0 to 2.5 | Minimum desirable. |
| C. M. Draper, City Engr., Lafayette, Ind. | 3.3 | 1.8 | 1.3 | Minimum sought. |
| Burgis Coy, City Engr., Fort Collins, Colo. | 3.0 | 2.5 | 2.0 | Minimum. |
| A. Sargent, City Engr., La Grange, Ga. | 2.5 | 1.5 | 1.0 | Minimum advisable when flushing with automatic tanks. |
| John L. Mason, Asst. City Engr., Modesto, Calif. | 2.0 | 1.2 | 1.0 | Minimum "All pipes of these sizes are usually flowing at least half-full during daily peak flows." |
| S. S. Crawford, Supt. of Sewers, Lake Charles, Fla. | 2.5 | 2.0 | 1.5 | Minimum desirable. |

TABLE 7

CONSENSUS OF OPINION REGARDING MINIMUM SLOPES REQUIRED FOR 8-INCH SEWERS

| Slope, ft. per 1000 ft. | Number of engineers recommending its use for minimum | Remarks |
|---|---|---|
| 6.2 | 1 | Normal minimum. |
| 6.0 | 1 | Vol. 3 ft. per sec. |
| 10.0 | 1 | Minimum recommended. |
| 5.0 | 1 | Minimum recommended. |
| 5.0 | 1 | Good practice. |
| 4.5 | 1 | Vol. 1.75 ft. per sec. $n$=0.015. |
| 4.0 | 17 | Minimum. |
| 3.3 | 1 | Minimum sought. |
| 3.0 | 1 | Minimum. |
| 2.6 | 1 | Automatic in flushing "2 ft. per sec." |
| 2.5 | 2 | Minimum advisable. |
| 2.5 | 1 | Minimum advisable with automatic flushing. |
| 2.5 | 1 | Absolute minimum, some trouble with sewer at this grade. |
| 2.0 | 1 | Slope to use in extreme case. |
| 2.0 | 1 | Minimum. |

ALEXANDER POTTER stated that his general practice was to lay all sewers at grades giving a velocity, when half full, of at least 2 ft. per second and preferably 2½ ft. With grades giving velocities much less than 2 ft. per second when half full, flushing and frequent cleaning are necessary. In order to avoid pumping or costly construction, however, Potter has used very flat grades at times. At Harrison, New York, about 5000 ft. of 20-in. sewer were laid with a fall of only 0.11 per cent. As the average flow will never more than quarter fill the pipe, arrangement has been made to flush it automatically once a day. At Kingsville, Texas, in order to avoid pumping, sewers flushed automatically once a day have been laid on grades as low as 0.1 per cent for 18-in. and 15-in., 0.15 per cent for 12-in., 0.2 per cent for 10-in., and 0.33 per cent for 8-in. In the southern part of Texas where the land is very flat many 8-in. sewers have been laid with a fall of only

0.20 per cent. In Corpus Christi, Texas, Potter found that practically all 8-in. laterals had been laid with a minimum grade of 0.2 per cent, and were kept clean by frequent flushing.

The Authors' practice, Metcalf & Eddy, 1928, is to endeavor to secure a velocity of at least 2.0 ft. per second in separate sewers and of 2.5 ft. per second in combined sewers, using $n = 0.015$ for pipe sewers 24 in. and smaller. There are instances, as noted in connection with the Worcester data, where lower velocities may be permitted if there is a high velocity immediately above or below. The actual velocity in the flatter length of sewer may then be higher than the computed velocity based upon the slope. All such cases would require special study before permitting a variation from the above limiting velocities.

## APPENDIX 3

### Experience With Flat Sewer Grades

The following material has been abstracted from correspondence received in answer to a form letter inquiry sent out by the Committee:

A. S. Jones, Design Engineer, Des Moines, Iowa

In cases where it has been impossible to serve properties except with very flat grades more than average maintenance costs have been experienced. According to information furnished by the foreman of the maintenance crews, the following sewers require frequent cleaning:

All 6-inch sewers regardless of grade on 8-inch sewer laid on a grade of 2.9 feet per thousand feet. Seven 10-inch sewers laid on grades ranging from 1.2 to 2.0 feet per thousand feet. One 10-inch sewer laid on a grade of 2.8 feet per thousand feet. Six 12-inch sewers laid on grades from 1.2 to 2.2 feet per thousand feet.

"The fact that one ten-inch sewer on a 0.28% grade, and one twelve-inch sewer on a 0.22% grade, each having a theoretical velocity of two (2) feet per second, require frequent cleaning, probably doesn't mean a thing in a system 350 miles long, except as an indication that two feet per second may be close to the minimum scouring velocity for pipe sewers."

HAROLD L. BRIGHAM, SUPERINTENDENT, WATER AND SEWAGE COMM., MARLBOROUGH, MASS.

"Street Sewers, 6″ to 10″ minimum grade .32 foot fall in 100 ft. We have grades such as this that have been in use since 1912 and they have required no more attention than any of our other street mains that have a much larger fall.

"Trunk Lines, 12″ fall 1′ in 500′, have been in use since 1917 with no trouble.

"Trunk Lines, 18″ diameter, 6750′ long, fall 1′ in 500 ft. in use since 1890 give no trouble except the first 450 ft. There are three street mains entering the trunk line at its beginning and considerable sand accumulates in the first section of 450′ which has to be scraped about three times as often as the rest of the line.

"Trunk Line, 20″ diameter, 11,500 ft. long, fall 1′ in 750′ has been in use since 1890 and gives no trouble."

BEN S. MORROW, CITY ENGINEER, PORTLAND, OREGON

"Below is a list of some of the sewers in Portland, Oregon, in which trouble is caused by flat gradients.

| | |
|---|---|
| S.E. Water Avenue | 8″ sewer, 1% slope |
| S.E. Yamhill Street | 8″ and 10″ sewer, less than 1% slope |
| N.W. Yeon Avenue | 12″ sewer, 0.12% slope |

(This sewer was installed for street drainage only. It becomes clogged with sand and street debris.)

| | | |
|---|---|---|
| N.E. 15th Avenue | 14″ sewer, | 0.2% slope |
| N.E. 16th Avenue | 14″ " | , 0.2% " |
| S.E. Birch Street | 8″ " | , about 1% slope |
| N. Overlook Boulevard | | 0.5% slope |
| S.E. 35th Avenue | | 0.8% " |
| S.E. 28th Avenue | | 1.23% " |
| S.E. 18th Avenue | 8″ pipe, | 0.66% " |
| S.E. 20th Avenue | 8″ " | 0.7% " |
| S.E. 21st Avenue | 10″ " | |
| N. Rodney Avenue | | 0.6% " |
| S.E. 36th Avenue | | 0.4% " |
| S.E. 35th Avenue | | 0.8% " |
| S.E. Elliott Avenue | | 0.35% " |

"Practically all of the sewers in Portland carry both sanitary and storm water sewage. As a result, during dry weather there is only a small percentage of the pipe capacity in use and the flow is far below the maximum velocity. In general, where these sewers are designed to have a velocity of 3 ft. per second, when flowing full or half full, they give little trouble, being cleansed by the storm water flow.

"Roots are troublesome when they can get through to the pipe joints. They seem to give more trouble in flat sewers than the steeper ones."

H. A. Jorgensen, City Engineer, San Diego, California

"Some trouble has been experienced in a separate system with sewers laid in slopes to give a velocity of 1.5 feet per second using $n = .013$ in the Kutter formula. Sewers laid at grades to give a velocity of 2 feet per second have not caused trouble. Some trouble has been experienced with deposits in trunk sewers laid at minimum grades when hillside laterals on steep grades discharge into them.

Stanley Shupe, City Engineer, Kitchener, Ontario

"Our experience in minimum sewer grades is that two feet per second velocity is the absolute minimum from which to escape periodic flushing problems.

"We have, however, the following examples in domestic sewer lines which operate satisfactorily, namely:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 9″ | Vitrified | Sewer | .33% | Grade | or | 4″ | per | 100′ |
| 12″ | " | " | .25% | " | " | 3″ | " | " |
| 15″ | " | " | .17% | " | " | 2″ | " | " |

"We have completed an expenditure of $1000 recently in cleaning the 48″ section with the .2 grade for six blocks downstream following the admission of industrial wastes from tanneries, municipal gas works and other industries. The principal ingredients of the deposits were cinders with sticky coating. The deposit was as much as 20″ deep and finally petering out and the remainder of the 48″ pipe at .2 and at .167 grade was self-cleansing. Accumulations occurred over a seven year period."

F. C. FROEHDE, CITY ENGINEER, POMONA, CALIFORNIA

"We do not have any sewer lines laid on a grade less than 0.26% grade. We place Flush Tanks at the head of each Lateral Sewer Line and cause Flush Tanks to operate more often than on Sewer Lines of greater grade. We have had no difficulty in keeping them clean. One small (12-inch) Outfall line was laid on a 0.33% grade, but it has never given us any trouble. This line was laid in 1901 and is still in operation.

"It is desirable not to have a grade of less than 0.25% on all sewer lines in sizes from 8 to 24 inches."

R. H. CASON, CITY ENGINEER, TAMPA, FLORIDA

"On account of the flat terrain in this section minimum grades are absolutely necessary to accomplish anything. As a general thing the grades here are lower than those recommended by authorities.

"We have had very little trouble from deposits in sewers due to flat slopes and low velocities except at the summit of some of the laterals.

"Some particular sections of sewers have to be rodded and flushed two or three times a year to remove objectionable deposits."

O. J. SEMMES, JR., CITY ENGINEER, PENSACOLA, FLORIDA

"(1) We have trouble from deposits in a number of sewers, nearly all of which were laid in 1909-1911. I cannot attribute this trouble to flat grades *per se*. There is in each case some other complicating factor such as the invert grade falling well below the hydraulic gradient as controlled by the tide level, poorly made joints that permit much silt and sand to enter the sewers. One common error is in providing too large sewers without providing adequate flushing facilities. It is sometimes necessary to provide a sewer that for some time will be too large to provide adequate velocities. In that case we would recommend the use of some type of flush tank. When use built up sufficiently high quantities to maintain adequate velocities the tank could be abandoned.

"(2) A sewer that is well constructed with good joints and no back pressure should stay in good condition if flushed once or twice a year. Sewers with grades that give velocities of less than 2 ft. per second should be flushed every 90 days. However, we have sewers that require flushing far more often, but these are defective in some way other than from flat grades.

"We have an 18″ line two miles long with $s = 0.0008$ that has never had a choke in its four years of use."

J. A. Peters, Superintendent of Sewers, Gloucester, Massachusetts

"Trouble is experienced continuously with an 8-inch sewer laid at a slope of 0.002.

"Flushing is necessary every two months on an 8-inch sewer laid at a slope of .003. Seven 8-inch sewers laid at a slope of 0.003, two laid at a slope of 0.004 and one at 0.005 require flushing every three months. A twelve-inch sewer laid at a slope of 0.002 requires flushing every month. The main intercepting sewer 21 to 36 inches in diameter and laid at slopes to give a velocity of about 2 ft. per sec. flowing full gives some trouble from deposits of gravel entering the sewer from old surface drains."

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The following material has been extracted from the returns to a round table discussion conducted at our request by the Editor of *Sewage Works Engineering*. The principal contribution to this discussion appeared in the March, 1941 issue with supplementary discussions appearing in the September, 1941 issue.

Watson Walker, City Engineer, Macon, Georgia

"Very little trouble has been experienced at Macon from deposits in sewers, due to flat slopes and velocities. The lines are flushed out twice a week to clear objectionable deposits. The following minimum slopes seem advisable: 8-inch—0.40%, for 12-inch—0.20%, and 15-inch—0.16%."

A. Sargent, City Engineer, La Grange, Georgia

"No trouble has been experienced at La Grange from deposits in sewers, attributable to flat slopes and low velocities.

"The following minimum slopes considered advisable, when flushing with automatic tanks: 8-inch—0.25%, 12-inch—0.15%, and 15-inch—0.10%."

E. D. Stapp, City Engineer, Davenport, Iowa

"We have experienced trouble with deposits in sewers due to flat slopes and low velocities, and we have also had trouble at the dead ends of sewers. One of the troublesome points is where we change from a steep to a flatter grade. All such changes are made at manholes, so that we can clean out the deposits, either by flushing or by removing.

"Our sewer department has a list of the places where the greatest difficulty is experienced, and these are flushed out about twice a year.

"We recommend a minimum grade of one-half of one per cent, although we have laid some sewers at flatter grades."

C. R. McAnlis, City Engineer, Fort Wayne, Indiana

"Some trouble has been encountered, due to the fact that our terrain, at points, is quite level, making flat grades necessary.

"Lines, where deposits are troublesome, are flushed out every six months.

"The following minimum slopes are recommended: 8-inch—0.5%, 10-inch—0.4%, 12-inch—0.32%, and 15-inch—0.15%."

V. B. Clark, City Engineer, Ansonia, Connecticut

"We have experienced trouble from deposits in sewers, due to flat slopes and low velocities. Troublesome lines are flushed out about once a month.

"On 8-inch pipe, we have used a grade of 0.3% and the results were satisfactory where there was considerable use of the line. In other cases, with a small flow, a grade of 0.5% has given trouble. I would say a minimum grade of 0.4% should be satisfactory under average conditions.

"For a 12-inch pipe, 0.25%, and for 15-inch pipe, 0.15% grades are recommended."

D. Y. McDowell, Water and Sewer Superintendent, Malvern, Arkansas

"No trouble from deposits in sewers due to flat slopes and velocities has been experienced in Malvern.

"All sewer lines in Malvern below fifteen inches have considerable fall. Some fifteen-inch lines, laid on 2/10 of 1% grade, have been in service for seven years and have given no trouble."

Charles Merle Jones, Superintendent of Sewers, Colorado Springs, Colorado

"In the opinion of the Colorado Springs Superintendent, lines should be cleaned and roots cut three or four times yearly. Lines should be inspected daily and flushed each week, or more often if the demand on them is great."

Lyle Payton, City Engineer, Stockton, California

"Stockton is a valley city, lying on the edge of San Joaquin River delta. Its terrain is very flat. Ground elevations, U.S.G.S. datum, vary from four feet to sixteen feet above sea level. Ground slopes are not more than four feet per mile. The sewage plants, lying about a mile outside of the city, are on land at sea level elevation, and surrounded by protecting levees.

"Flushing is considered to be essential to keep a sewerage system with flat gradients in good running order, yet due to curtailed maintenance funds, Stockton has reduced the number of operating automatic flushers from 1054 in 1929 to 61 in 1934, thus reducing yearly water service costs from $8500 to $710.

"The first step in this reduction of active flush tanks was to discontinue the use of flushers on lines having a computed velocity of more than 1.5 feet per second. This reduced the number of operating flushers to 308, and the cost to $3545 in the first year.

"The maintenance crew then by experiments over the next four years reduced the number to 61 in 1934. The savings thus effected amounts to $48.70 per mile of sanitary sewer.

"We have not experienced any material increase in the number of sewer stoppages, as is attested by the fact that the maintenance crew was reduced by one man in 1931 and he has not been replaced to date.

"There are two districts, where sewers were extended beyond the limits for which they were designed, which require weekly inspection, but no unusual trouble has resulted from discontinuing flushing.

"The number of stoppages in house branches, between the main and the property line, amount to 400 per year in round numbers, and in main and lateral sewers, to about 100 per year.

"The history of the last ten years of sewerage system stoppages indicates that they are the result of failure to comply with the existing law and common sense.

"A minimum velocity of two feet per second is recommended."

RICHARD G. MANLEY, CITY ENGINEER, UPLAND, CALIFORNIA

"We have not had trouble with stoppages. Lines are flushed once a year.

"We have 8-inch pipe with slope as low as .22%, with flush tanks. Without flush tanks, the minimum slope should not be less than .43% for 9-inch pipe, .23% for 12-inch pipe, and .16% for 15-inch pipe, even if there is a good depth of flow."

BURGIS COY, CITY ENGINEER AND SUPERINTENDENT OF STREETS, FORT COLLINS, COLORADO

"Deposits in sewers here have been attributed to flat slopes and low velocities. Lines are flushed about every three months; oftener, if they are long lines."

W. E. FORD, SUPERINTENDENT OF STREETS AND SEWERS, HUNTINGTON PARK, CALIFORNIA

"We have had difficulty with deposits in sewers due to flat slopes and low velocities.

"Our flush tanks are set to operate every 24 hours. Where we have no flush tanks, we use our street flushers once a month."

HOWARD CARTER, CITY ENGINEER, SANTA MONICA, CALIFORNIA

"No trouble has been experienced with deposits in sewers due to flat slopes and low velocities.

"I should say the number of house connections would determine how often lines should be flushed out to clear objectionable deposits."

L. R. PARMELEE, CITY ENGINEER, HELENA, ARKANSAS

"Flat slopes and velocities have not caused deposits in sewers here.

"We flush manually every thirty days, generally, and more often on lines susceptible to trouble, which is usually traceable to grease."

CLAUDE M. DRAPER, CITY ENGINEER, LAFAYETTE, INDIANA

"We have experienced trouble with flat grades in sewers. The ones we have had trouble with have had to be cleaned at least once a year."

W. W. Mathews, Superintendent, Gary Sanitary District, Gary, Indiana

"We have not had trouble with deposits in sewers. The main interceptors have been in service only about five months, so sufficient time has not elapsed to determine whether there will be silting troubles with Gary sand.

"The time interval for flushing lines will vary with various locations, soil conditions and size and grades of sewers so that no definite time intervals for flushings can be given."

D. L. Bundy, City Engineer, Redendo Beach, California

"Deposits in our sewers, due to flat slopes and low velocities, make it necessary to flush such lines every two weeks."

Arthur N. Beck, Assistant Sanitary Engineer, Bureau of Sanitation, State Department of Health, Montgomery, Alabama

"On numbers of occasions, while in the field, municipal authorities have called to our attention stoppage of sewers. Trouble, it appears, usually occurs on lines with flat slopes and on those serving only a small population. In many cases, periodic flushing is not practiced, and is done only when the sewers become clogged. Particularly is this true in our small towns."

W. O. Castello, Assistant Engineer, Sacramento, California

"We have very little trouble with stoppages due to flat slopes and slow velocity. Most of our sewer trouble results from tree roots."

Frank E. Alderman, City Engineer, South Gate, California

"We have not been troubled with deposits in sewers due to flat slopes and low velocities.

"Objectionable deposits are removed from sewers by running a ball through, every six to twelve months, when we flush."

W. L. Popp, City Engineer, San Jose, California

"The only trouble we have experienced with deposits in sewers has been where settlement has taken place. We flush certain sewers about twice during the summer period."

JOHN L. MASON, ASSISTANT CITY ENGINEER, MODESTO, CALIFORNIA

"We have experienced trouble with deposits in sewers in areas where only a few houses are connected to a sewer which has a capacity for the entire area.

"Flushing should be employed at irregular intervals, as the depositing appears. We do not recommend regular intervals for flushing, but do inspect and flush all collecting sewers once a year if they need it. We flush with hose from fire hydrants."

G. S. HINCKLEY, CITY ENGINEER AND SUPERINTENDENT OF STREETS AND SEWER DISPOSAL, REDLANDS, CALIFORNIA

"Some trouble has been experienced here due to flat slopes and low velocities.

"Flushing the lines about every three months seems necessary.

"We have 8-inch and 12-inch sewers on slopes of 2 feet to 4 feet per mile."

S. S. CRAWFORD, SUPERINTENDENT OF SEWERS, LAKE CHARLES, LOUISIANA

"Flat slopes and low velocities have caused deposits in our sewers.

"Each time it rains, such lines must be flushed out. Here, this means about twice a month."

C. C. BONEBRAKE, CITY ENGINEER, ORANGE, CALIFORNIA

"We have had trouble in sewers, caused by deposits, especially near wash racks in garages.

"We try to drag all our small sewer lines at least once a year. I regard a 3-inch fall per 100 feet, i.e., .25% as a minimum for 8-inch lines. However, sometimes even this cannot be had. A 2-inch fall per 100 feet is desirable on larger lines."

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The following material has been compiled from the technical literature.

Extracts from Symposium on "Experience with Flat Sewer Grades" printed in *California Sewage Works Association Journal,* Vol. 2, No. 1, 1929.

ANDREW M. JENSEN, CITY ENGINEER, FRESNO, CALIFORNIA

"The minimum grades in Fresno, for the smaller size sewers, are as low as 0.15 ft. per 100 ft. of length, for a 6-inch sewer. We also have a good deal of 0.20 ft. per 100 ft. on 6-in. sewers. Most of these flat grades are at the extreme ends—the beginning of the 6-in. lateral lines around the exterior boundaries of the city—and we have almost no trouble with them. The only time we have trouble with these flat sewers is when there are only two houses connected in one block. We have no odor trouble, no discharge trouble, and we get along very well with them. We flush out some of these dead ends occasionally when the maintenance crew makes its rounds through the city.

"Some of the cities nearby have flatter grades than we have. In Sanger there is a good deal of 6-in. lateral sewer on a 0.1 per cent grade; also 8-in. sewer with 0.05 ft. per 100 ft. which is extremely flat and Sanger has no particular trouble with its sewer system.

"Clovis has 0.2 ft. per 100 ft. on 6-in., 0.2 ft. on 8-in., and 0.05 ft. on 10-in.

"In Fresno we use a small self-propelling nozzle, to clean some of the smaller sewers occasionally. We have only one crew, consisting of two men, to clean and maintain the entire sewer system. They do nothing else and are able to take care of all the sewers. There are no automatic flush tanks at the end of these lines."

F. J. ROSSI, CITY ENGINEER, MODESTO, CALIFORNIA

"Sewer 8 in. to 24 in. in diameter laid on slopes to give a velocity of 1.7 ft. per sec. assuming $n = 0.013$ in the Kutter formula gives good results.

"My experience shows that when a sewer becomes sluggish it builds up a head creating enough velocity to clean itself. Modesto has many sewers of 6-in. and 8-in. diameter, laid on grades as flat as 0.15 ft. per 100 ft., and there have been few cases of serious trouble. The reason for this may be that the water services in Modesto are only 4% metered; and the per capita consumption of water averages 212 gal. per day. This, no doubt, admits more water into the sewers, thus diluting the sewage and allowing an easier flow for the solids.

"Most all cases of stoppage can be traced back to lines poorly laid, tree roots, grease from restaurants or similar establishments, sand getting into the line from street drains, or people being careless in emptying heavy paper and rags into the slop sinks.

"Stoppage occurs most any place in the line, but more often in some line in a new subdivision with one or two houses only in a block. I take care of the latter condition by installing a 2-in. water line in the end of the sewer and flushing every three months.

"Our most serious case of stoppage happened in a 6-in. line laid on a curve. It was necessary to construct a manhole at the center of the curve in order that the sewer could be rodded from both directions. It took two days to open up this line. An engineer should never attempt to put a small sewer line on a curve.

"All sewers from 6 in. to 12 in. diameter, are rodded and flushed once a year, and only in exceptional cases is it necessary to clean them oftener. Sewers requiring cleaning more frequently are those which receive an exceptional amount of grease, or sewers troubled with roots in the line.

"We flush 38 miles of sewers from 6-in. to 12-in. diameter at an approximate cost of $8.00 per mile. This is done by rodding and flushing with a hose, using city hydrant pressure.

"The most serious cases of odors are due to sluggish sewers and dead end lines with few houses in the block. Flush-tanks placed on the end of such lines give good results, but on account of their cost, I have used instead 2-in. water lines from the main to end of sewer, and flush at three-month intervals.

"No doubt the odor is caused by not having enough service connections to the main to maintain a steady flow of water to carry away the solids. It is probable that septic action has started in such cases.

"I should say that it is much cheaper to clean sewers on flat grades occasionally than attempt to lay sewers on grades that will give sufficient velocity to keep the line clear without cleaning or flushing.

"In cities of flat areas, it is hardly possible to lay lands of any great length on steep grades without some being too near the surface. To have a self-cleaning system, it would be necessary to have automatic pumping plants to raise the sewage from sumps into the main outfall sewer. The expense of such a system would be prohibitive, especially to a small municipality."

L. M. BERRYHILL, TULARE, CALIFORNIA

"Mr. Jensen has covered the situation at Tulare in describing that at Fresno. Maximum grade is 0.3 ft. per 100 ft. with practically no trouble. Men from the street department take care of necessary work."

L. F. BARZELLOTTI, LODI, CALIFORNIA

"The lowest grades I know of are at Lodi, where maximum is 0.1 ft. per 100 ft. and minimum is less than ½ in. One man takes acre of all the sewers. All storm water is separated, and domestic sewage flows better on a flat grade than on a very steep one because it consists of water carrying light floating materials which gradually move along at low velocities, but if the current is swift it slaps grease against the sides of the joints which remains and eventually produces an obstruction. Except for sticks or overalls thrown in, there have been no objections in Lodi sewers.

"Lodi has 28 miles of sewers running generally from east to west, and natural fall being in that direction. Sewer elevations are determined by the topography and the location of Lodi, which is 50 miles from the tide water as the crow flies. At present sewage effluent is discharged to an irrigation canal which eventually drains to tide water, and if it becomes necessary in the future to run sewage to tide water, a sufficient grade can be obtained. This condition has resulted in some sewers being built on grades as low as ½ in. per 100 feet."

A. M. RAWN, ASSISTANT CHIEF ENGINEER, LOS ANGELES COUNTY SANITARY DISTRICT, CALIFORNIA SEWAGE WORKS JOURNAL, VOL. 2, NO. 1, 1929

"The city and county of Los Angeles use for the design of their large systems a grade which will produce a minimum velocity of 2 ft. per sec. with the pipe flowing full or half full. Grades producing less than this velocity are termed by the engineers of this section, flat grades.

"The carrying ability of water is such that it moves solids along by dragging, by temporary or intermittent suspension, or by continuous suspension; and by one or all of these means the solids are transported in a sewer pipe. The transporting power of water may be increased by an increase in depth, the velocity remaining unchanged. These attributes of water have been coupled with sedimentation experiments from time to time, to produce the criterion of 2 ft. minimum designed velocity for sanitary sewers to prevent sedimentation in the pipe.

"No doubt the results are clouded somewhat by storm sewage and street drainage, and it can be said with certainty that 2 ft. per sec. designed flow is not a definitely proven figure. It can be stated with equal certainty that it affords a safe minimum velocity and one which will occur under practically all conditions of contribution to a large sewer system in a metropolitan district.

"In spite of the fact that sharp grades are considered advisable, conditions under which the system is built, that is, size, topography, the method of treatment, and the amount immediately available for expenditure, will influence the engineer, and no doubt have done so, to depart from prescribed practice and use flat grades. The fact that these flat grade lines are nearly all successful in operation is due, in my opinion, as much to the size of the system and the relative location of the sewers as to any other feature. I believe that endless trouble would result were systems the size of those in Los Angeles County, to be built using uniformly flat grades and no pumping stations. A great many reasons could be given for this, but it is almost sufficient to say that the magnitude of a project materially influences its maintenance, and while a small organization might keep its finger very closely on the pulse of a small system, a larger organization would have a great deal more difficulty keeping close contact with a much larger system. Trouble would undoubtedly be the result."

## APPENDIX 4

### Theoretical Considerations

The primary function of a sanitary sewer is to transport the solid matter in the sewage, the water in the sewage serving as the carrier. It is of importance therefore that sewers be designed to control the flow of the water in *such* a manner as to facilitate transportation of the suspended solids.

Sediment is transported by flowing water in three ways:

(1) Bed Load Movement. The solid particles are dragged or rolled along the bottom of the pipe by the shearing or *tractive force* of the water.

(2) Suspended Load Movement. The solid particles are continuously enveloped by flowing water and are transported in *suspension.*

(3) *Saltation.* The solid particles move alternately in *traction* and *suspension.*

Solids which are transported in suspension move with the velocity of the water, whereas solids moved in traction progress along the sewer at much lower velocity. In order to insure freshness at the point of disposal it is obviously desirable that the sewage solids be transported mainly in suspension.

Most of the suspended sewage solids are heavier than water and will settle. Some of the solids such as grease and particles containing gas are lighter than water and rise. There is thus a vertical motion of the suspended particles due to their specific gravity which if not compensated for will result ultimately in clarification of the water with all heavy suspended matter on the bottom and all light suspended matter at the surface. If the flow in the sewer were *laminar* (viscous or streamline), clarification would take place, and the sewer would be ineffective. The flow in sewers is *turbulent,* however, and the accompanying eddies tend to throw the particles back into suspension.

The modern theory of fluid turbulence in circular pipes flowing full has thrown a great deal of light upon the mechanism of turbulent flow and upon the effect of pipe roughness on friction loss. The theory has not yet been adapted quantitatively to open channels, but the phenomena are similar. Numerous investigations have been made in recent years on the transportation of sediment by flowing streams. These studies have for the most part been related to rivers, but the principles involved apply equally well to sewers. The more recent of these studies have sought to correlate transportation phenomena with the turbulent flow theory, and have met with considerable success.

## Theory of Turbulent Flow in Full Pipes

In a pipe containing a liquid flowing uniformly, the pressure intensity varies statically across any normal section of the stream.

That is, the pressure head plus the potential head (the piezometric head) is constant over the cross section. Between any two sections distance $L$ apart, there is a loss of piezometric head due to friction. If the pipe flows full, there is, corresponding to this lost head, a difference in pressure on the bases of any concentric cylinder within the pipe which is exactly counterbalanced by the shear on the outer surface of the fluid cylinder, as follows:

$$(p_1 - p_2)\ \pi r^2 = 2\pi\ r\ L\ \tau \text{ - - - - - - - - - - - - - - - - } (1)$$

in which

$p_1$ and $p_2$ = the pressure intensities on the bases of the fluid cylinder,
$r$ = the radius of the fluid cylinder,
$L$ = the length of the cylinder,
and $\tau$ = the intensity of the shearing stress or tractive force on the surface of the fluid cylinder.

The above relation is valid for both viscous and turbulent flow. It will be noted that the shearing stress, $\tau$, equals $\frac{p_1 - p_2}{L} \cdot \frac{r}{2}$. It is therefore zero at the pipe center and increases uniformly to a maximum value at the walls of the pipe. At the walls of the pipe, the shearing stress is

$$T = \frac{p_1 - p_2}{L}\ \frac{D}{4} = w\,\frac{h}{L}\,\frac{D}{4} \text{ - - - - - - - - - - - - - - - } (2)$$

where $D$ = the pipe diameter,
$\frac{h}{L}$ = the slope of the hydraulic grade line,
and $w$ = the specific weight or weight per unit volume of the liquid.

For an open channel, there is a shearing stress on the wetted perimeter and none at the free surface. The shearing stress is variable around the wetted perimeter, but an equation analogous to (2) may be written for the average shear by substituting for $D$ in (2) its equivalent in terms of the hydraulic radius, $4R$, as follows:

$$T = \frac{p_1 - p_2}{L}\,R = w\,\frac{h}{L}\,R \text{ - - - - - - - - - - - - - - } (2a)$$

For a rectangular channel of infinite width the hydraulic radius is equal to the depth. Equation (2a) for this case is the well-known formula of du Boys (1879) for the tractive force on the bed of a stream.

By dimensional analysis it may be shown that

$$T = \frac{f}{4} \rho \frac{V^2}{2} \text{ - - - - - - - - - - - - - - - - - - - - - - } (3)$$

in which $f$ = the friction factor in the Weisbach-Darcy friction formula,

$\rho$ = the mass density of the liquid $= \frac{w}{g}$, where $g$ = the acceleration of gravity,

and $V$ = the mean velocity of the fluid, the discharge per unit area of cross section.

By equating the values of $T$ in (2) and (3), the familiar Weisbach-Darcy friction formula is obtained:

$$h = \frac{fL}{D} \frac{V^2}{2g} = \frac{fL}{4R} \frac{V^2}{2g} \text{ - - - - - - - - - - - - - - - - - } (4)$$

American engineers engaged in sewer design are accustomed to the use of the Chezy formula with Kutter's formula for the Chezy coefficient $C$. Kutter's formula for $C$ is a complex empirical equation involving a roughness coefficient $n$, the values of which most American engineers have become familiar with. A much simpler formula for the Chezy $C$ by Manning involves a similar roughness coefficient $n$, whose values are substantially equal to those of Kutter's $n$. Because of the simplicity of the Manning formula, it is gradually replacing the Kutter formula for open channel hydraulic computations. The value of Manning's roughness factor $n$ and the Chezy coefficient $C$ in terms of the more general factor $f$ are as follows:

$$n = 1.486\, R^{\frac{1}{6}} \sqrt{\frac{f}{8g}} \text{ - - - - - - - - - - - - - - - - - - - } (5)$$

$$\text{and } C = \sqrt{\frac{8g}{f}} \text{ - - - - - - - - - - - - - - - - - - - - - - - } (5a)$$

The friction factor $f$ is dimensionless, whereas the Manning coefficient

$n$ as shown in (5) is influenced by the hydraulic radius.

A great many experiments have been made to evaluate the coefficient $f$ for pipes flowing full. Among the most reliable are those of Nikuradse (1932-33) shown in Fig. 4. In this figure, $f$ is plotted

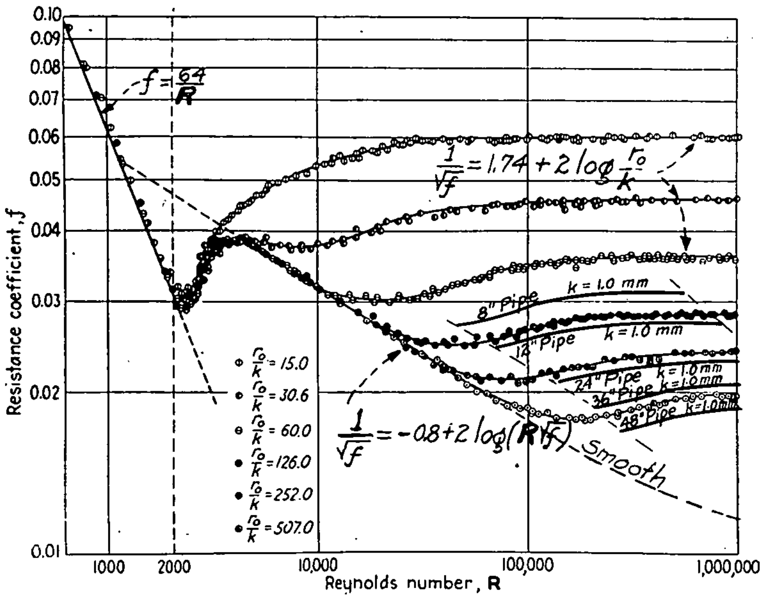


FIG. 4.

against the Reynolds number **R** which is also non-dimensional. The Reynolds number

$$\mathbf{R} = \frac{VD}{\nu} = \frac{4VR}{\nu} \text{ - - - - - - - - - - - - - - - - - - } (6)$$

in which $\nu$ = the kinematic viscosity, having the dimensions $\frac{(\text{length})^2}{\text{time}}$,

$= \frac{\mu}{\rho}$, where $\mu$ is the absolute coefficient of viscosity having the dimensions $\frac{\text{force} \times \text{time}}{(\text{length})^2}$

For water and sewage, the value of $\mu$ and hence of $\nu$ depends almost entirely upon the temperature. Table 8 gives values of $\mu$ and $\nu$ in

TABLE 8

| Temp., degrees F. | 35 | 40 | 50 | 60 | 70 | 80 | 90 |
|---|---|---|---|---|---|---|---|
| $\mu$, $\frac{\text{pound seconds}}{\text{square foot}} \times 10^5$ | 3.55 | 3.23 | 2.74 | 2.35 | 2.05 | 1.793 | 1.587 |
| $\nu$, $\frac{\text{square feet}}{\text{second}} \times 10^5$ | 1.83 | 1.665 | 1.407 | 1.21 | 1.056 | 0.924 | 0.817 |

the English system. The apparent viscosity of sewage sludge is greater than that of water and depends upon the solids content.

It will be noted on Fig. 4 that for values of Reynolds number less than about 2000, $f$ is $\frac{64}{\mathbf{R}}$. This is the region of laminar or viscous flow which is characterized by the Poiseuille law of flow through capillary tubes. In viscous flow each fluid lamina slides over the adjacent one in such a way that the shearing stress,

$$\tau = -\mu \frac{dv}{dr} \quad (7)$$

where $\frac{dv}{dr}$ = the space rate of change of velocity in the pipe cross section, having a value of zero at the center and a maximum value at the pipe walls.

It may be seen from (7) that the absolute viscosity $\mu$ is merely a proportionality constant and that it must have the dimensions stated above. If equation (7) is integrated from the center to the wall of the pipe and the velocity at the wall is taken as zero, it will be found that the velocity distribution is parabolic across the pipe diameter as shown in Curve 1 of Fig. 5. The maximum velocity in the center is twice the mean velocity $V$, and the head loss is

$$h = \frac{32\nu}{g}\frac{L}{D^2}V = \frac{64}{\mathbf{R}}\frac{L}{D}\frac{V^2}{2g} \quad (8)$$

This is the equation of Poiseuille. Its validity is attested by the fact that it has been used for more than a hundred years for viscosity measurements with capillary tubes. One of the outstanding features of this law is that the head loss varies directly as the mean velocity.

In turbulent flow, the main body of the fluid away from the walls is in eddy motion as well as translation. There is thus a turbulent

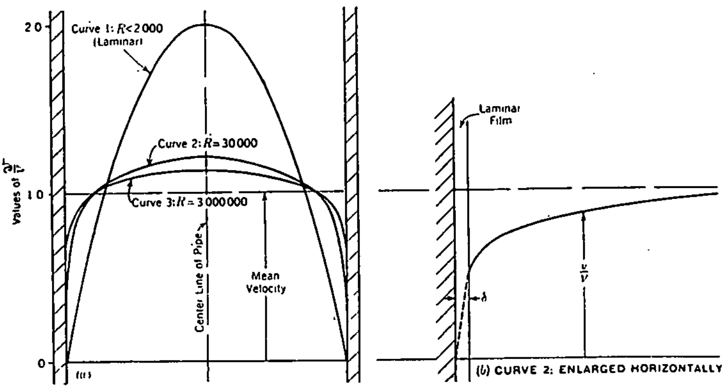


FIG. 5.—VELOCITY DISTRIBUTION IN A VERY SMOOTH PIPE.

exchange of fluid and momentum in a lateral direction which tends to equalize the velocity distribution. This effect is shown in Curves 2 and 3 of Fig. 5. The higher the value of the Reynolds number, the greater is the turbulent exchange and the more nearly does the mean velocity $V$ approach the value of the peak velocity at the center of the pipe. Measurements of the velocity distribution in turbulent flow have seemed to indicate that the velocity at the wall is not zero. Nevertheless, theoretical considerations show that the velocity at a smooth wall should be zero; and Prandtl developed the theory of the laminar boundary layer to account for the high shearing stress at the wall.

Boussinesq (1877) showed that the shear intensity in purely turbulent flow may, as with viscous flow, be expressed as a function of the velocity gradient. In a pipe where both viscous and turbulent shear are present,

$$\tau = \mu \frac{dv}{dy} + \eta \frac{dv}{dy} \qquad (9)$$

in which $\frac{dv}{dy} = -\frac{dv}{dr}$ of (7), and

$y =$ the lateral distance from the wall of the pipe. Near the pipe wall all the shear is viscous and the second term is zero. Hence the velocity gradient $\frac{dv}{dy}$ at the wall must be very high to account for the high shearing stress in turbulent flow. In the main body of the fluid away from the wall, the velocity gradient is so small that the first term of equation (9) is negligible. The proportionality factor $\eta$ for turbulent shear is not a constant for a given fluid at a definite temperature, as is the viscosity factor $\mu$. The value of $\eta$ depends upon the fluid density, the dynamic quality of the turbulent process and upon the point in the stream being considered.

In turbulent flow the velocity $v$ at any point in the cross section is actually fluctuating continuously because of the superimposed eddies. $v$ must be taken as the temporal mean velocity at a point. The superimposed eddies will impart a secondary motion which may be considered to have the mean temporal velocity components $v'_x$ in the direction of the pipe axis and $v'_y$ in the transverse direction. These mean velocities must be considered without regard to sign, since the actual mean of positive and negative fluctuations is zero. Osborne Reynolds (1894) has shown that the term $\eta \dfrac{dv}{dy}$ of equation (9) may be expressed as a function of the density of the fluid and the momentary fluctuations of velocity.

Prandtl (1925) introduced a linear dimension $l$ called the *mixing length* which is analogous to the mean free path in the kinetic theory of gases. The mixing length is the average transverse distance at any point in the flow over which a small fluid mass is carried by the eddies of turbulence. It is thus a measure of the size of the eddies, and $v'_y$ is a measure of the velocity of transport laterally. Prandtl then showed that

$$\eta = \rho\, v'_y\, l = \rho\, l^2 \frac{dv}{dy} \qquad (10)$$

The quantity $\dfrac{\eta}{\rho} = \epsilon$ is called the kinematic turbulence factor or the kinematic eddy viscosity. This factor is a direct measure of the transporting capacity of the mixing process, and is equally applicable to transport of momentum, vorticity, salinity, sediment or heat.

In order to proceed further with the development of the turbulent flow theory, von Kármán (1930) found a relation between the mixing length $l$ and the velocity gradient $\dfrac{dv}{dy}$. This relation will not be given here. It had previously been shown by Stanton (1911) that the form of the velocity distribution curve in the central region of uniform turbulent flow is independent of the velocity gradient at the boundary for the same boundary shear. It will be noted from (3) that $\frac{1}{\sqrt{f}}$ is proportional to $\frac{V}{\sqrt{\frac{T}{\rho}}}$. The term $\sqrt{\frac{T}{\rho}}$ has the dimension of a velocity and is called the *friction velocity*. Hence the dimensionless ratio of mean velocity to friction velocity varies inversely as the square root of $f$ for both viscous and turbulent flow. In accordance with Stanton's findings the *velocity defect* $v_{max} - v$ at any point in the cross section is directly proportional to the friction velocity and independent of the conditions at the wall for both rough and smooth pipe. Using this relation von Kármán was able to combine equations (3), (9) and (10) for the turbulent region producing an expression for the velocity distribution which does not contain the mixing length. The constants in the von Kármán expression were evaluated for smooth

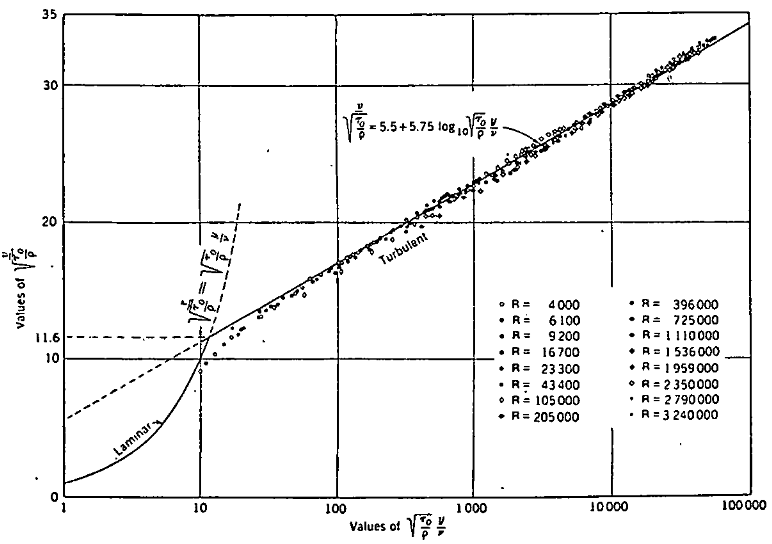


FIG. 6.—UNIVERSAL VELOCITY DISTRIBUTION FOR SMOOTH PIPES.

pipe by means of the experiments of Nikuradse with the following results:

$$\frac{v}{\sqrt{\frac{T}{\rho}}} = 5.5 + 5.75 \log \left( \sqrt{\frac{T}{\rho}} \frac{y}{\nu} \right) \qquad (11)$$

This equation is shown in Fig. 6 plotted to a semi-logarithmic scale ($T$ is shown $\tau_o$ on the figure).

Equation (11) is applicable to the turbulent region in a smooth pipe. In the very thin laminar film near the wall the velocity may be assumed to vary directly with the distance from the wall and $\tau$ may be taken as equal to $T$, such that $v = \frac{Ty}{\mu}$. Hence in the boundary layer,

$$\frac{v}{\sqrt{\frac{T}{\rho}}} = \sqrt{\frac{T}{\rho}} \frac{y}{\nu} \qquad (12)$$

There is probably a smooth transition between the boundary layer and the turbulent region, as is indicated by the experimental points on Fig. 6 in the region of the intersection of the two curves. Nevertheless the intersection of the two curves represented by equations (11) and (12) may be taken as a measure of the thickness of the boundary film. The value of $y$ obtained by solving these two equations simultaneously is thus equal to the thickness of the laminar film,

$$\delta = 11.6 \frac{\nu}{\sqrt{\frac{T}{\rho}}} = \frac{11.6\nu}{V\sqrt{\frac{f}{8}}} \qquad (13)$$

It will be noted that the thickness of $\delta$ of the boundary film is independent of the pipe size and depends only on the mean velocity, the friction factor and the viscosity. For sewers with a temperature range from 35° to 80° F., a velocity range from 1 to 12 feet per second and a value of $f$ between 0.02 and 0.03, the thickness of the laminar film will be between 0.14 mm. and 4.1 mm., assuming of course that the wall is perfectly smooth. In a sewer or other commercial pipe, the wall is not perfectly smooth. There are projections due to roughness or sediment. If the magnitude $k$ of these projections is small compared to the thickness $\delta$ of the boundary layer, as shown in Fig. 7a, the wall irregularities are fully enclosed in the boundary

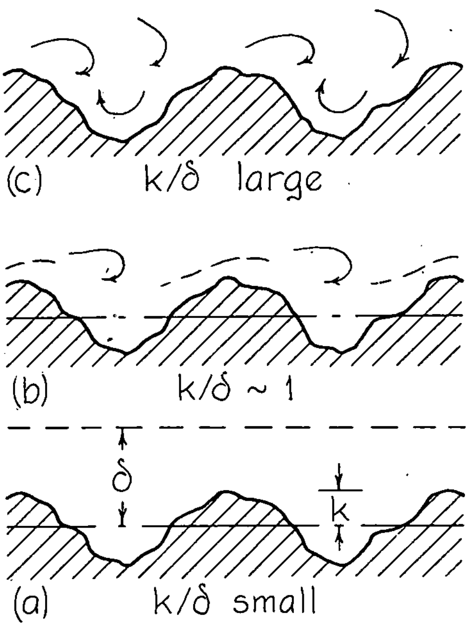


FIG. 7.—DEVELOPMENT OF TURBULENCE AT A ROUGH BOUNDARY.

layer and the flow is the same as with a smooth wall. With moderate roughness, $k$ equal about 0.6 $\delta$, the irregularities begin to influence the flow outside the laminar film. With increasing roughness turbulence at the wall becomes greater until with $k$ about 10 times the value of $\delta$ turbulence is fully developed at the wall. Such a pipe is called a rough pipe.

Nikuradse's research included work with pipes artificially roughened by means of sand grains cemented to the pipe walls. The grain diameter was used to represent the roughness dimension $k$. The constants in von Kármán's general expression for rough pipe were evaluated by means of the Nikuradse data to yield the following expression for the velocity distribution:

$$\frac{v}{\sqrt{\frac{T}{\rho}}} = 8.48 + 5.75 \log \frac{Y}{k} \qquad (14)$$

This equation is shown plotted on Fig. 8 with the Nikuradse experi-

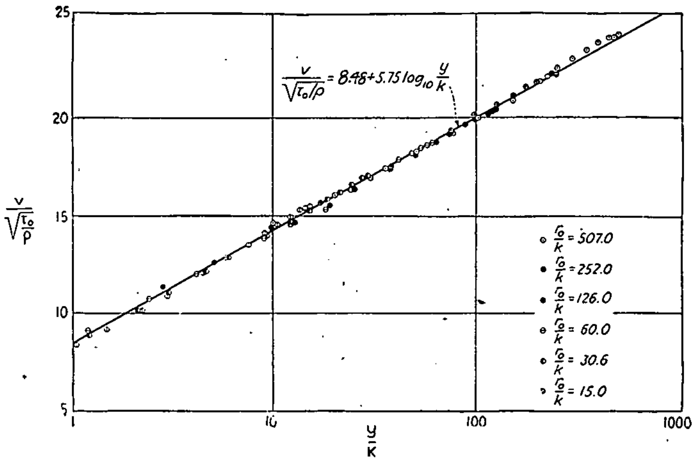


FIG. 8.—UNIVERSAL VELOCITY DISTRIBUTION FOR ROUGH PIPES.

mental data. The value $r_o$ shown on this figure and on Fig. 4 is the pipe radius $\frac{D}{2}$. The relative roughness is thus shown by the ratio $\frac{r_o}{k}$

Equations (11) and (14) for the velocity distribution in smooth and rough pipes respectively may be transformed to give expressions for the mean velocity $V$. Also, since from (3) $\frac{V}{\sqrt{\frac{T}{\rho}}} = \sqrt{\frac{8}{f}}$, further transformations will yield expressions for the friction factor $f$ in terms of the variables of equations (11) and (14). These mathematical developments are omitted here, but the results are as follows:

For smooth pipe

$$\frac{1}{\sqrt{f}} = -0.8 + 2 \log (\mathbf{R} \sqrt{f}) \qquad (15)$$

For rough pipe,

$$\frac{1}{\sqrt{f}} = 1.74 + 2 \log \frac{r_o}{k} \qquad (16)$$

It will be noted that the value of $f$ for smooth pipe depends only on the Reynolds number **R** and for fully rough pipe only on the relative roughness $\frac{r_o}{k}$. This is clearly shown in Fig. 4.

The type of roughness studied by Nikuradse is homogenous. The irregularities on the walls of a sewer are not uniform in size and spacing and the joints moreover present another form of roughness not covered by Nikuradse's studies. A single factor such as $k$ is probably not sufficient to account for all types of roughness, although an *effective* value of $k$ may be determined experimentally for a particular surface by a method developed by Schlichting. This has not yet been done for sewers.

Some light upon the effective value of $k$ may be had by assuming a value and computing the corresponding value of Manning's $n$ for sewer pipe. For example, with $k = 1.0$ mm. = 0.04 inches, $\frac{r_o}{k} = 100$, 150, 300, 450 and 600 for 8, 12, 24, 36 and 48-inch pipe respectively. With a mean velocity ranging from 1 to 12 feet per second and a temperature of 50° F., the Reynolds number ranges from 47,500 to 570,000 for 8-inch pipe and from 285,000 to 3,400,000 for 48-inch pipe. Curves representing the values of $f$ within the corresponding ranges of **R** and for $k = 1$ mm. have been plotted on Fig. 4 for 8, 12, 24, 36 and 48-inch pipe flowing full. The average values of $f$ and the corresponding values of Manning's $n$ computed from equation (5) are given in the table below:

VALUES OF $f$ AND MANNING'S $n$ FOR $k = 1$ mm.

| Pipe Size, Inches | 8 | 12 | 24 | 36 | 48 |
|---|---|---|---|---|---|
| Friction factor $f$ | 0.029 | 0.026 | 0.022 | 0.020 | 0.018 |
| Manning's $n$ | 0.0117 | 0.0118 | 0.0122 | 0.0125 | 0.0124 |

Each 20 to 30% increase in the value of $f$ or 10 to 14% increase in the value of $n$ corresponds with a 100% increase in the effective value of $k$. Hence a value of $n = 0.015$ sometimes used for the design of small sewers corresponds with a value of $k$ of about 1/8 inch (3 mm.), whereas the value of $n = 0.013$ which is generally used for the design of sewers corresponds with an effective value of $k$ of about 1/16 inch. These values of $n$ imply greater values of $k$ than actually exist, as will be shown in the following paragraph. The value of $n$ for partially full pipes, however, is greater than for full pipes; and since it is common practice in design to assume the same value of $n$ for all depths of flow the values used are not out of line with actual conditions.

The actual value of $f$ as measured by Wilcox (Bulletin No. 27, University of Washington Eng. Exp. Station, 1924) for 8-inch vitrified clay sewer pipe flowing full was about 0.019 (Manning's $n = 0.0095$); and for concrete sewer pipe flowing full was about 0.022 (Manning's $n = 0.0102$). These pipes were clean and were carefully laid with good joints. The experiments were conducted with water. A similar set of experiments on clay and concrete drain tile was made by Yarnell and Woodward (Bulletin No. 854, U. S. Dept. of Agriculture, 1920). The tile was laid carefully in earth but with open-butt joints. Sizes tested included 4, 5, 6, 8, 10 and 12-inch. The values of the friction factors for the pipes flowing full were about the same for concrete and clay. There was a tendency towards a decrease in the value of $f$ with increase in slope. The average values were as follows:

| Pipe size inches | 4 | 5 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|
| Friction factor $f$ | 0.033 | 0.028 | 0.027 | 0.025 | 0.023 | 0.021 |
| Manning's $n$ | 0.0111 | 0.0106 | 0.0107 | 0.0108 | 0.0108 | 0.0107 |

A number of investigators have made measurements of Kutter's $n$ in sewers in actual use (see Metcalf and Eddy, "American Sewerage Practice," Vol. I, 1928, pg. 82 to 87). Most of the data are for sewers flowing partially full. The data indicate that the friction factor is greater for partially full pipes than for pipes flowing full, but the data are too meagre to effect a correlation.

## Turbulent Flow in Partially Filled Pipes

The theory of turbulent flow has not as yet been extended to partially full pipes, the usual case with sewers. At the free water surface there is no shear. All the shearing resistance is distributed around the wetted perimeter, and since the cross section is not circular a non-uniform distribution of the shear is to be expected. The unit shearing stress at the bottom will exceed the average value $T$ as computed from equation (2a). For partially full pipes, moreover, the peak velocity is no longer at the center of the pipe, and the velocity distribution is correspondingly affected. The net result of these influences appears to be an increase in the value of the friction factor for partially full pipes.

It is common practice in sewer design to assume a constant value of the friction factor regardless of the depth of flow. The experiments of Wilcox and of Yarnell and Woodward indicate that this practice may result in considerable error. A study of the ratios of the values of $f$ for partially full and full pipes indicates a trend as shown in Fig. 1. This curve was plotted from the average results of *824* experiments by Yarnell and Woodward on well-laid drain tile, both concrete and clay. The average values of $\frac{f}{f_{\text{full}}}$ for each pipe size for slopes from 0.0005 to 0.015 are shown in the following table:

VALUES OF $\frac{f}{f_{\text{full}}}$

| Pipe size, inches | 4 | 5 | 6 | 8 | 10 | 12 | Average |
|---|---|---|---|---|---|---|---|
| Depth ratio $\frac{d}{D} = 0.8$ | 1.28 | 1.15 | 1.22 | 1.18 | 1.06 | 1.21 | 1.20 |
| " " " $= 0.6$ | 1.8 | 1.34 | 1.50 | 1.38 | 1.12 | 1.40 | 1.42 |
| " " " $= 0.4$ | — | 1.9 | 1.9 | 1.7 | 1.32 | 1.70 | 1.70 |

The values of the ratios $\frac{f}{f_{\text{full}}}$ from these experiments as well as those of Wilcox were quite scattered; but the averages for each ratio of $\frac{d}{D}$ checked closely for both studies, the maximum deviation being about five per cent. The experimental data seemed to show no consistent influence of the pipe material or the variation in pipe size or slope. Wilcox used 8-inch pipe on slopes of 0.005 to 0.04, so that the range of slopes for both sets of data was 80 to 1. Since the variation in friction factor with depth as represented by Fig. 1 is based on the average results of only two experimental investigations, and since the individual values from which the averages were obtained deviated considerably, the curves of Fig. 1 are not to be considered as precise. More experimental work, with more precise control, is urgently needed. Nevertheless the use of Fig. 1 for sewer design will yield more reliable results than are obtained by current practice.

In Fig. 1, the *hydraulic elements* for the partially full pipe computed with the variable value of $f$ are shown in solid lines. For comparison the corresponding ratios, assuming a constant value of Kutter's $n$, are indicated with broken lines.

## Bed Load Movement and Critical Tractive Force

In Nikuradse's experiments with full pipes artificially roughened by cementing sand grains on the inner surface, it was found that the velocity distribution could be expressed as follows:

$$\frac{v}{\sqrt{\frac{T}{\rho}}} = \varphi\left(\frac{d}{\delta}\right) + 5.75 \log \frac{y}{d} \quad \text{-------------} \quad (17)$$

in which the grain diameter $d$ is used as a measure of absolute roughness $k$, and $\varphi\left(\frac{d}{\delta}\right)$ is a function of the ratio of grain size to boundary layer thickness. Measured data plotted in the form $\frac{v}{\sqrt{\frac{T}{\rho}}} - 5.75 \log \frac{y}{d}$ versus $\frac{d}{\delta}$ as shown in Fig. 9 indicate the validity of equation (17). The term $\varphi\left(\frac{d}{\delta}\right)$ becomes 8.48 for rough pipe ($\frac{d}{\delta}$ greater than about 20), and equation (17) is the same as (14).

Since the artificial roughness used by Nikuradse had a texture similar to that of leveled sand beds, the foregoing relations should be applicable to the study of the beginning of bed-load movement. Shields reasoned that the initial movement of material of any given particle size $d$ would require the existence of a certain critical velocity $v_c$ at a distance above the bed proportional to the particle diameter, $y_c = \alpha_1 d$. Hence from (17),

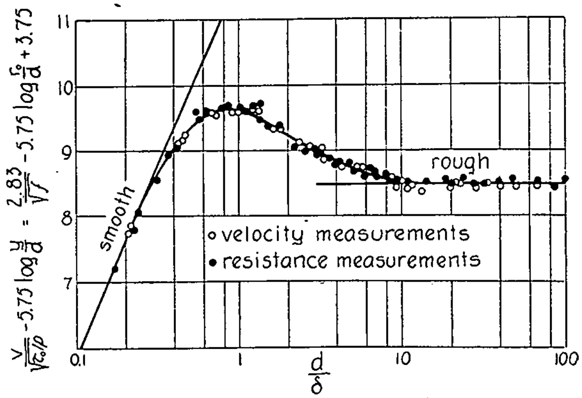


FIG. 9.—INFLUENCE OF BOUNDARY-LAYER THICKNESS UPON VELOCITY DISTRIBUTION AND RESISTANCE FOR ROUGH BOUNDARIES.

$$v_o = \sqrt{\frac{T}{\rho}} \left[ 5.75 \log \alpha_1 + \varphi \left( \frac{d}{\delta} \right) \right] = \sqrt{\frac{T}{\rho}}\, \varphi_1 \left( \alpha_1, \frac{d}{\delta} \right) \qquad (18)$$

The drag force of the liquid past the particle tending to move it is given by Newton's law as follows:

$$F_D = C_D\, d^2\, \frac{\rho v_c^2}{2} \qquad (19)$$

in which $C_D$ is a coefficient whose magnitude depends on the shape of the particle and the Reynolds number of the flow conditions around the particle. Hence

$$C_D = \varphi_2 \left( \alpha_2, \frac{v_c d}{\nu} \right) \qquad (20)$$

where $\alpha_2$ is the shape factor. Now the Reynolds number $\frac{v_c d}{\nu}$ may be expressed from (18) as follows:

$$\frac{v_c d'}{\nu} = \sqrt{\frac{T}{\rho}}\, \frac{d}{\nu}\, \varphi_1 \left( \alpha_1, \frac{d}{\delta} \right) \qquad (21)$$

and since from (13) $\frac{\sqrt{\frac{T}{\rho}}}{\nu}$ is proportional to $\frac{1}{\delta}$,

$$\frac{v_c d}{\nu} = \varphi_3 \left(\alpha_1, \frac{d}{\delta}\right) \qquad (21a)$$

and $$C_D = \varphi_4 \left(\alpha_1, \alpha_2, \frac{d}{\delta}\right) \qquad (20a)$$

Substituting this value of $C_D$ and the value of $v_c$ from (18) in equation (19), the drag force becomes

$$F_D = \varphi_4 \left(\alpha_1, \alpha_2, \frac{d}{\delta}\right) \frac{d^2\rho}{2} \frac{T}{\rho} \left[\varphi_1 \left(\alpha_1, \frac{d}{\delta}\right)\right]^2$$

$$= \frac{Td^2}{2} \cdot \varphi_5 \left(\alpha_1, \alpha_2, \frac{d}{\delta}\right) \qquad (19a)$$

The resistance to motion of the particle is proportional to its effective weight and the form and arrangement of the particles:

$$F_R = \alpha_3 \; (w_s - w) \; d^3 \qquad (22)$$

in which $w_s$ = the weight per unit volume of the solid particle,
and $\alpha_3$ = the coefficient of friction due to the form and arrangement of the particles on the bottom.

At the beginning of motion the two forces $F_D$ and $F_R$ must be equal. Hence

$$\frac{T_c d^2}{2} \varphi_5 \left(\alpha_1, \alpha_2, \frac{d}{\delta}\right) = \alpha_3 \; (w_s - w) \; d^3 \qquad (23)$$

in which $T_c$ is the *critical tractive force* required to produce motion of the particles along the bottom. Since $\alpha_1$, $\alpha_2$ and $\alpha_3$ are dependent upon the size, shape and arrangement of the particles of size *d*, the equation may be simplified as follows:

$$\frac{T_c}{(w_s - w) \; d} = \varphi \left(\frac{d}{\delta}\right) \qquad (23a)$$

The results of experiments by Shields and others on the critical tractive force of uni-granular materials of varying specific gravity are plotted in Fig. 10 in accordance with equation (23a). The similarity of the trend of the curve in Fig. 10 to the velocity distribution in rough pipes (Fig. 9) is apparent.

For particles to be moved along the bottom, it is evident from Fig. 10 that a value of $\dfrac{T_c}{(w_s - w) \; d} > 0.04$ is required to start motion;

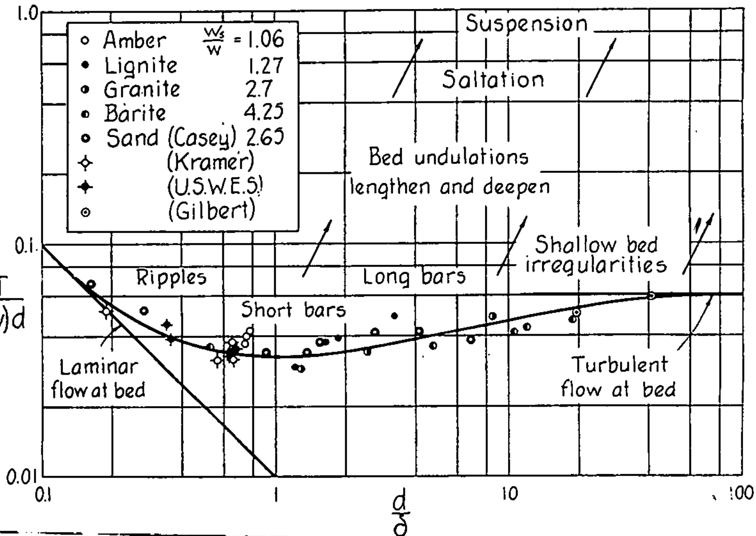


FIG. 10.—BEGINNING OF BED-LOAD MOVEMENT AS A FUNCTION OF GRAIN DIAMETER AND BOUNDARY-LAYER THICKNESS.

and that a value of $\dfrac{T}{(w_s - w)\, d} = 0.8$ is adequate to keep some of the material in suspension. If we use this ratio as a measure of the minimum velocity in sewers, then, from equation (3):

$$T = \frac{f}{8}\, \rho\, V^2 = \beta\ (w_s - w)\, d \qquad (24)$$

from which

$$V = \sqrt{\frac{8\beta}{f}\, g\, \frac{w_s - w}{w}\, d} \qquad (25)$$

where $\beta = 0.04$ to start motion and 0.08 for adequate cleansing.

This relation indicates that the velocity required to transport material is independent of pipe size and depth of flow; and that it depends only on the friction factor and the particle size and specific weight. Consider a friction factor $f = 0.025$ for a pipe flowing full. Then from equation (25) the size of sand (sp. gr. = 2.65) which will be transported effectively ($\beta = 0.8$) at 2 fps velocity is 0.09 mm. and the size of organic material (sp. gr. taken arbitrarily at 1.2) is 0.75

mm. Material nearly 20 times these sizes, if not sticky, will be barely moved along the sewer invert.

Now consider the same pipe with the depth of flow, for example, taken at 1/5 the pipe diameter. From Fig. 1, $f = 2.1 \times 0.025 = 0.052$, and $V = 0.48 \times 2 = 0.96$ fps. Then from equation (25) the material transported is only 48% as large as computed above for the full pipe. It is evident from this example that fluctuations in flow must be considered in establishing minimum slopes for sewers. Consider now a velocity of 2 fps with the depth of flow 1/5 the diameter. In this case, since $f = 0.052$, the material transported is 2.1 times the size moved with the pipe full at the same velocity. The velocity required to keep a partially full pipe clean, therefore, is less than that for the full pipe; but it is greater than the velocity which will obtain at low flows if the slope of the pipe is fixed to obtain adequate cleansing with the pipe flowing full.

For equal cleansing effect at all depths of flow, it is obvious from equation (24) that $T$ or $fV^2$ must be the same for all depths. Hence the ratio $\frac{V}{V_{full}}$ for equal cleansing effect is equal to $\frac{1}{\sqrt{\frac{f}{f_{full}}}}$. This ratio has been plotted as one of the *hydraulic elements* on Fig. 1. The utility of this curve to the sewer designer may be illustrated as follows. Suppose it is found that a velocity of, say, 2.0 fps is required in a certain pipe when it flows full to keep it clean. Suppose also that the fluctuations in discharge will be such that at minimum flow the depth will be 20% of the pipe diameter. Then from Fig. 1 the velocity at this depth should be $0.7 \times 2.0 = 1.4$ fps for adequate cleansing. The slope of the sewer should therefore be sufficient to produce this velocity at 0.2 full or a velocity when full of $\frac{1.4}{0.48} = 2.92$ fps. If the sewer is laid at the required slope its cleansing action will be more than adequate at flows greater than the minimum.

Equations (24) and (25) throw no light upon the quantity of material which may be transported in a sewer, nor upon the relative amounts on the bottom and in suspension. The equations indicate the velocity required for a given state of motion. For example, to start motion on the bottom; or, for a given concentration of solid material, a state of motion such that a certain proportion is in suspension. There are experimental data to indicate that a change in amount of solid matter in transport will produce a corresponding change in the relative amount being carried in suspension; but these data are too meagre to be of use quantitatively.

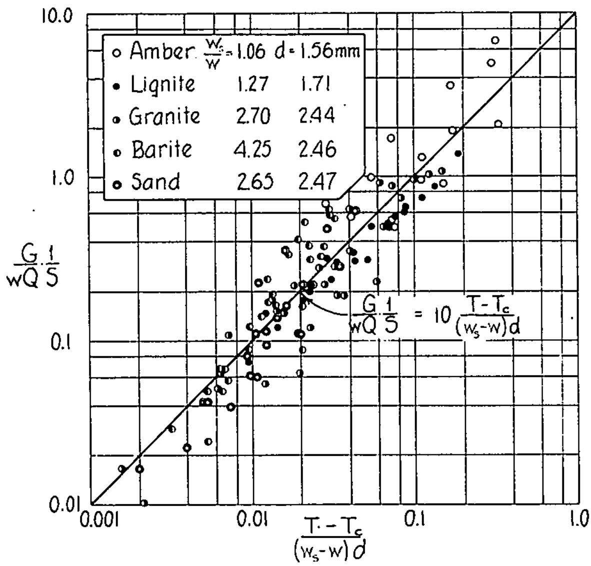


FIG. 11.—TRANSPORTATION OF BED-LOAD.

From an analysis of measured data, Shields found that the quantity of material being transported along the bottom could be expressed by the following relation as shown in Fig. 11:

$$\frac{G}{wQ}\,\frac{1}{S} = 10\,\frac{T - T_c}{(w_s - w)\,d} \qquad (26)$$

where $S$ is the hydraulic slope, $G$ is the weight (measured under water) of sediment transported and $wQ$ is the weight of water flowing. The experimental data are for bed-load movement and some saltation. Whether this relation will hold for the quantity of material transported when a considerable proportion is in suspension is not known. Since the actual weight of material transported $W = \frac{w_s}{w_s - w} G$, the slope $S = \frac{f}{8g} \frac{V^2}{R}$ from (4) and $T - T_c = \frac{w}{8g} (fV^2 - f_c V_c^2)$ from (3); it follows from (26) that the concentration of material transported in ppm is

$$C = 10^7 \left(\frac{1}{8g}\right)^2 \frac{1}{R} \frac{s}{(s-1)^2 d} fV^2(fV^2 - f_c V_c^2) \quad \text{- - -} \quad (27)$$

in which $V_c$ = the velocity required to start motion,
$f_c$ = the friction factor at the start of motion,
and $s$ = the specific gravity of the material moved.
It also follows that the weight of material moved per second is

$$W = 10 \left(\frac{1}{8g}\right)^2 \frac{A}{R} \frac{s}{(s-1)^2 d} wfV^3 (fV^2 - f_c V_c^2) \quad \text{- -} \quad (28)$$

in which $A$ = the area of the cross section. If the velocity $V$ is large compared to $V_c$ and $f$ is substantially equal to $f_c$, it will be noted that the concentration of sediment transported is nearly proportional to the fourth power of the velocity and the weight of material moved is nearly proportional to the fifth power of the velocity.

Consider, for example, a 12-inch sewer flowing full at 2 fps velocity. If $f = 0.025$, this sewer will effectively transport sand ($s = 2.65$) 0.09 mm. in size and organic matter ($s = 1.2$) 0.75 mm. in particle size. The velocity required to start motion on the bottom (assuming $f_c = f$) is, from equation (25), equal to 0.45 fps. Then at 2 fps velocity, the concentration of sand which can be transported is 19,500 ppm and of organic matter is 72,200 ppm, computed from (27). Also the weight transported is 1.91 lbs. of sand per second and 7.07 lbs. of organic matter per second. Note that the submerged weight of both materials is the same, but the actual weight of the organic matter is greater because of its lighter specific gravity. The transporting power of the water is a function of the submerged weight of the material.

## Transportation in Suspension

Particles which are in motion along the bottom by traction may be deflected upwards into the stream. Such particles may fall back in saltating motion or they may be carried higher into the main body of the stream in suspended motion. The direction of motion of a particle thrown up into the liquid will be determined by the vector sum of its own settling velocity and the velocity of the liquid surrounding the particle. Since in turbulent flow the eddies produce vertical components of velocity, a particle may momentarily be carried either upward or downward. The temporal mean of the vertical velocities in the stream is zero, however, while the settling velocity of the particle is constant and continuous in one vertical direction. Therefore transportation in suspension is not of itself an equilibrium condition. If all particles were removed as soon as they reach the bottom, the stream would eventually contain no suspended matter. The suspended load must be continuously replenished from the bottom in order for the concentration of suspended matter to remain constant.

If the sediment transportation of a stream as a whole is in equilibrium, the rate of upward transfer due to turbulence must equal the rate of downward motion due to settling. Since the upward motion of the fluid masses due to the mixing process is equal to the downward motion, it follows that, for a uniform concentration of suspended matter, there will be no net vertical movement of sediment due to turbulence. For the mixing process to effect upward transfer of sediment, it is necessary that the eddies carry more material upward than they carry downward. This requires a concentration gradient with the eddies carrying more material from regions of high concentration than they carry from regions of low concentration. Settling tends to clear the stream from the top thus developing a concentration gradient; and the eddies tend to move the material in the direction of decreased concentration or upwards.

The upward transfer is related to the mixing velocity $v'_y$ and the mixing length $l$. If at any depth $y$ from the bottom the concentration of suspended particles per unit volume is $c$ and the concentration gradient is $-\frac{dc}{dy}$, the rate of upward transfer per unit area is equal to the rate of upward flow $v'_y$ times the difference in concentration $-l\,\frac{dc}{dy}$. For particles having any settling velocity $v_s$, the rate of downward motion per unit area due to settling is $cv_s$. Hence at equilibrium,

$$c\,v_s = -v'_y\,l\,\frac{dc}{dy} \quad \text{-------------------} \quad (29)$$

Integration of (29) yields an expression for the relative concentration of suspended matter at any distance $y - a$ above some arbitrary reference level $a$, as follows:

$$\ln\frac{c}{c_a} = -v_s\int_a^y \frac{dy}{v'_y l} = -v_s\int_a^y \frac{dy}{\epsilon} \quad \text{-------} \quad (30)$$

or

$$\frac{c}{c_a} = e^{-v_s\int_a^y \frac{dy}{\epsilon}} \quad \text{-------------------} \quad (30a)$$

in which $e$ is the base of the natural system of logarithms and $\epsilon$ is the kinematic eddy viscosity or *exchange (Austausch) coefficient.*

It will be noted from (9) and (10) that for fully turbulent flow

$$\epsilon = v'_y\,l = \frac{\frac{\tau}{\rho}}{\frac{dv}{dy}} \quad \text{-------------------} \quad (31)$$

The value of $\epsilon$ at a point in a stream thus depends upon the shear and the velocity gradient. The shear is readily evaluated in terms of the tractive force $T$ or the mean velocity $V$. Various attempts have been made to evaluate $v'_y$, $l$ and $\epsilon$, but the theory is inadequate in that it gives a zero value of $\epsilon$ at the boundaries and at the center of a full pipe. Eddies are known to travel across the center of a full pipe and to produce a jetting action on the bed of a stream. The integral in (30a) may be evaluated for a full pipe by means of von Kármán's theory.

The validity of equation (30) has been demonstrated experimentally in jars in which uniform mixing was accomplished with an agitator throughout most of the depth (i.e., $\epsilon$ = a constant). It probably is valid in a flowing stream throughout most of the depth. The absolute value of either $c$ or $c_a$ cannot be determined from (30), and no means are at hand yet for estimating the magnitude or concentration of the suspended load.

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8. Separate Systems of Sewerage, Staley & Pierson, 2d Edit., 1891, Ihling Bros. & Everard. "Velocity Required to Prevent Deposits," p. 96.

9. Stockton Maintenance Problems, Lyle Payton, California Sewage Works Journal, Vol. 13, No. 1, 1941.
10. Methods and Problems of Sewer Maintenance, R. F. Brown, California Sewage Works Journal, Vol. 12, No. 2, 1940.
11. City Engineering at Its Best, E. J. Cleary, St. Petersburg, Fla. Practice—Eng. News-Record, V. 122, May 11, 1939, p. 669. Flat grades, 0.04% for a 48-in. interceptor.
12. Sewer system and modern sewage treatment plant for town of 500, built with W.P.A. (Village of Roscoe, N. Y.) Flat grades resulted in a minimum of 0.4 ft. per 100 for the 8-in. line. Public Works, March 1939, p. 16.
13. Flow in pipes containing bottom deposits, V. W. Willits, Public Works, Feb. 1939, p. 15. Gives graph for calculating flow in pipes containing sand, based on experience with low-velocity, gravity flow conduit 30 in. diameter on .001 slope, circular.
14. Constructing sewers for a town of 1500, Robert M. Lingo, Public Works, Feb. 1939, p. 12 (at Wilson, Kansas). Flat grades for 10-in. outfall, 0.20% used as minimum.
15. Water sewers and services (at the New York World's Fair), L. B. Roberts, Eng. News-Record, V. 121, Sept. 22, 1938, p. 375. Minimum velocities conform with N. Y. State Dept. Health requirements of 2 ft. per sec. (p. 377).
16. Sewer flushing—Considers question of grades, concludes insufficient depth of flow rather than velocity is responsible for silting. Gives curves of depths of flush with effects down-sewer. Recommends flushing for sewers at certain flat grades. Water Works and Sewerage, August 1937, v. 84, p. 294.
17. Steel pipe for trunk line sewer, Olney Borden, Public Works, July 1937, p. 13. (Outfall for Village of Liberty, N. Y.—18-in. on grade of 2.12%; 24-in. on 0.564% grade; and 30-inch on grade of 0.1732%.)
18. Sewer maintenance an engineering problem, Raymond R. Ribal, Civil Engineering, V. 7, March 1937, p. 209. (Practice is to attempt to obtain grades producing velocities of from 1.5 to 2 ft. per sec.)
19. Inverted siphon installation at Appleton, Wis., F. E. Fulmer, Public Works, Nov. 1936, p. 26 (gives velocities in siphon).
20. Friction coefficient of cast iron sewer after 30 years' service (tests of 36-in. c.i. pipe laid on 0.121% grade). Water Works and Sewerage, June 1934, V. 81, p. 220.
21. Cast iron pipe trunk sewers in Washington Suburban Sanitary District, Robert B. Morse. Water Works and Sewerage, V. 80, June 1933, p. 193. (Considers frictional resistance of concrete vs. cast iron pipe and gradients. 36-inch pipe sewers.)
22. Design and construction of sanitary sewers, Chester A. Smith, Municipal Sanitation, V. 3, March 1932, p. 112, p. 114. (Discusses "Alignment and grades.")
23. Tests on large (60-in.) concrete pipe sewer show Kutter's "*n*" to be 0.014—Elson T. Killam. Eng. News-Record, V. 107, Dec. 10, 1931, p. 935.
24. Sewer Grades, A. M. Rawn, California Sewage Works Journal, Vol. 2, Nov. 1, 1929.

25. Experiences with Flat Sewer Grades at Modesto, Fresno, Tulare and Lodi, California. California Sewage Works Journal, Vol. 11, No. 1, 1929.

26. Minimum sewer grades, Ernest McCullough. Eng. News-Record, V. 103, Aug. 8, 1929, p. 229. (Gives formulae for determining minimum sewer grades.)

27. On the proper value of Kutter's *"n"* for sewer computations, Charles W. Sherman, Eng. News-Record, V. 8, June 8, 1922, p. 948. (Summarizes the practice of many authorities and makes following recommendations: 0.013 for all concrete sewers, if well made with smooth interior surface, and unless allowance is made for drop of head at changes of direction in manhole, "it is probable that a higher value for *'n'* than 0.015 should be used in the design of pipe sewers.")

28. B.S.C.E. V. viii, 1921, p. 240. Report on inverted siphons for sewers, gives actual velocities and discusses hydraulic principles of siphons.

29. Sewer grades and sewage disposal—A British view on low sewer velocities. Abstract of address of A. J. Martin, Eng. News-Record, V. 87, Oct. 20, 1921, p. 638.

30. How sewers are designed in New York City, S. D. Bleich, Munic. and County Eng., V. LX, March 1921, p. 92. (P. 97 gives formula for loss of head in bends in excess of straight pipe, measured along tangents for a central angle of 90 deg. of $Kv$ in which $v$ is velocity of flow, ft. per sec., and $K$ a factor depending for its value on the radius of bend. For a radius of 5 ft., $K = .0023$; for 15 ft., .0048; for 25 ft., .0065; and for 40 ft., .0077. Gives various elements which determine grade.)

31. Some points to observe in design of sewerage systems and disposal works, Theodore Horton (N. Y. State Health Dept.), Mun. and County Eng., V. LIX, August 1920, p. 59. (Recommends minimum grades providing velocities equal to 2.0 ft. per sec. and in no case less than 1.5 ft. per sec. running full or one-half full.)

32. Design of Milwaukee's New Sewerage System Based on News—Conditions 35 Years Hence, by Carl H. Nordell, Eng. News-Record, April 12, 1917, p. 65. (P. 68. The interceptors are designed for a minimum velocity of 2.5 ft. per sec. when running full. When first placed in service the velocity will only be 2 ft. per sec. A brief discussion of the matter of minimum velocities.)

33. Building 22 miles of pipe sewers under one contract (at Topeka, Kans.), Eng. News, V. 75, March 30, 1916, p. 592.

| | |
|---|---|
| 8-in. pipe sewers | 0.32% min. grade |
| 10-in. | 0.24 |
| 15-18-21 in. | 0.16 |
| 45 in. | 0.1 |

34. Sewerage system for Panama-Pacific Exposition grounds at San Francisco, William C. Willard, Eng. Record, V. 70, Nov. 14, 1914, p. 538 (p. 539 gives table of minimum grades for pipe sewers 6 to 24 in. in size).

35. Rational methods and guesswork in sewer design. (Letter by H. P. Eddy, Eng. Record, V. 70, Oct. 31, 1914, p. 496 (". . . the size of the pipe must

be such as to provide self-cleansing velocities under usual conditions of flow . . ." with the sewers running full).

36. Am. Pub. Health Assoc., Report of Committee on the Collection and Disposal of Sewage—Abstracted Eng. News, V. 70, Oct. 16, 1913, p. 759—("Velocity of flow.—For self-cleaning velocities (ideal) gradients should not be less than 2 ft. per sec. for separate and 2½ ft. for combined systems. Lesser gradients in order to decrease trench excavation are often used. These require more frequent flushing but otherwise give no trouble provided a smooth interior surface is secured. Well constructed, reasonably flushed sewers, taking domestic sewage only, may have velocities as low as 1 ft. per sec.")
37. Rehabilitation of Hoboken sewerage system, Eng. Record, V. 67, Feb. 22, 1913, p. 221. (Peculiar conditions of flat grades and settlement. "These conditions, according to the report (of James H. Fuertes, consulting engineer) are not favorable for maintaining the sewers in a clean condition, for during the time when the velocity is checked down to less than 1½ ft. per second, deposits will take place in the sewers. Under some conditions these may become too solid to be flushed out again by the flowing sewage on the falling tide.")
38. Water supply and sewerage of the National Rifle Range, Camp Perry, Ohio, Eng. News, V. 65, June 22, 1911, p. 758. (Flat grades necessary; for 10" line 0.2% and for 6 in., 0.3%.)
39. Modern procedure in district sewer design, W. W. Horner (St. Louis), Eng. News, V. 64, Sept. 29, 1910, p. 326. (Grades which give velocities under 2 ft. per sec. should be avoided, and should never be used where there are no inlets at the upper end to flush the sewer.)
40. The sewer system of San Francisco and a solution of the storm-water flow problem, C. E. Grunsky, Trans., A.S.C.E., V. LXV, Dec. 1909, p. 294. (P. 389 gives desirability of a velocity of 3 ft. per sec. to prevent silting, and the values of "*n*" allowed for different materials. Discussions of Robert G. Dieck.)
41. The Sewerage System of New Orleans—I, Eng. Record, V. 53, May 26, 1906, p. 640. (On p. 642 there is a discussion of grades and velocities made necessary by the flat topography.)
42. The cleaning and flushing of sewers, J. L. Woodfall, and others, Jour. Assoc. of Engineering Societies, Oct. 1904, V. xxxiii, p. 199 (Discussion of self-cleansing velocities as found at several places in Massachusetts, with suggestions regarding flushing.).
43. The cleaning and flushing of sewers—abstract of an article by W. D. Hubbard before Boston Society of Civil Engineers, Mar. 2, 1904. (P. 341 gives 6-in. sewers, 1% grade; 8 and 10-in., 0.5% grade; 12-in., 0.15% grade. Min. grade for house connections (5-in.) is 2%. Flush tanks on the lower levels.)
44. The Shelby, Ohio, sewerage system—abstract—paper by J. B. Weddell before Ohio Soc. of Surv. and Civ. Engrs., Eng. Record, V. 43, Mar. 9, 1901, p. 228, flat topography, gives grades for 8 to 18 in. pipe sewers.

45. Maintenance of the system of separate sewers at Newton, Mass., Stephen Childs, Journal Assoc. Eng. Socs., 1899, 0.94. Minimum sewer is 8 in. and minimum rate for it 0.50 ft. per 100 ft. Trouble was caused by laying an 8-in. sewer at 0.40 ft. per 100. Discussion of T. Howard Barnes—at Medford, Mass., minimum size is 8-in. and minimum grade 0.50%. William Nelson of Laconia, N. H., gives minimum grade as $\frac{2.40}{d}$ with one 6-in. lateral at 0.26%.
46. Flushing in pipe sewers, H. N. Ogden, Trans. A.S.C.E., V. XI, Dec. 1898, p. 1, Sewer grades, with respect to flushing in pipe sewers.

CHARLES REED MAIN

Professional success and its accompaniment, the approbation and admiration of countless friends marked the career of Charles Reed Main.

Mr. Main was born in Lawrence, Feb. 10, 1885, the son of Charles T. Main and Elizabeth (Appleton) Main. In 1893 the family moved to Winchester where Mr. Main attended the public schools. His graduation from Dartmouth in 1906 was followed by three years at Massachusetts Institute of Technology, where he received the S.B. Degree in Mechanical Engineering.

His professional work began in California, on projects with Stone and Webster, and in Montana, with his father, covering a period of about six years. Then he returned to Boston and entered the office of his father, Charles T. Main, as an associate, later serving both as President and Treasurer of the corporation.

Mr. Main was especially interested in the Boston Society of Civil Engineers in which he became a member in 1909. He served on various committees and also elected a Director, served five years as Treasurer, one year as Vice-President and was President of the Society at the time of his death.

He was also a member of the American Society of Mechanical Engineers and of the Engineering Societies of New England. He was life secretary of his class of 1909 at the Massachusetts Institute of Technology and was for many years a member of the Engineers Club of Boston and of the Boston Rotary Club.

Through his interest in education he became a term member of the corporation of the Massachusetts Institute of Technology for five years and for the past several years served as a member of the Board of Overseers of the Thayer School of Civil Engineering at Dartmouth College.

He took part in the civic affairs of the town of Winchester, serving as a member of the Finance Committee from 1919 to 1922, as Selectman from 1922 to 1924, as a member of the Highland District School Building Committee in 1925 and of the Wyman School addition committee two years later. He was a member of the Board of Trustees of the Home for Aged People. A 32nd degree Mason in the Scottish Rite bodies, he held membership in William Parkman Lodge, A. F. and A. M. and in Aleppo Temple of the Mystic Shrine.

Mr. Main died August 22, 1942. He leaves a wife, the former Rose Frost of Santa Barbara, Calif., and two sons, Ensign Charles T. Main, 2nd, of Charleston, S. C., and Pvt. Samuel F. Main of New York. He also leaves his father, Charles T. Main, a sister, Alice A. Main, both of Winchester; and a brother, Theodore Main, of Holyoke.

## ADDITIONS

*Members*

FRANCIS J. CRANDALL, 24 Beverly Road, Wellesley, Mass.

## DEATHS

EDWARD G. GUSHEE, Dec. 4, 1941
CHARLES R. MAIN, Aug. 22, 1942
LEONARD E. SCHLEMN, 1941
DANIEL L. TURNER, Mar. 12, 1942

Cambridge, Mass.) Elementary School, Cambridge, Mass. Graduated in 1916 from Rindge Technical School, Cambridge, Mass. 1916-1918, 2-year structural engineering course at Wentworth Institute. Completed the following Engineering Defense Training courses: Structural Design and Detail, June, 1941; Advance Structural Design and Detail, January, 1942; Design of Concrete Buildings, May, 1942. Experience, 1916-1920, Boston & Maine Railroad, Structural draftsman, architectural division 2 years, bridge division 2 years; 1920-1922, F. A. Waldron, Consulting Engineer, New York City. Structural Engineering; 1922-1927, Mulhall & Holmes, Architects & Engineers, Boston, structural engineer; 1927-1936, J. A. Singarella, Inc., contractors, Boston, structural engineering, estimating and general superintendent, outside construction; 1936-1940, Watertown Arsenal, Civil Engineering, maintenance and new construction. 1940 to date, Jackson & Moreland, Engineers, Boston, designing engineering for Generating Stations. Hold a License for the Control of Building Operations. Class A.B.C. issues by the Boston Building Department, since 1924. Refers to *E. H. Cameron, W. W. Davis, W. D. Henderson, B. A. Rich, F. N. Weaver, F. S. Wells.*

JULIUS LASKER, Roxbury, Mass. (b. October 10, 1888, Smela, Russia.) Graduated from the Russian High School. In 1904 was a special student for the year at the Technical High School at Providence, R. I. Graduated from Brown University in 1909. Was awarded the Howell Premium. Experience, 1911-1917, with Boston & Maine Railroad, designed the Lynn viaduct and many other bridges; 1923-1925, with Stone & Webster in charge of the design and detailing of the steel for Weymouth Station of the Boston Edison Company and of Southern California Edison at Long Beach; 1925-1929, in Palestine as engineer in charge of construction for Mr. M. Shoolman of Boston; designed and supervised the construction of a large group of buildings at Harfa, was also in charge of construction of the buildings of the Hebrew University in Jerusalem; 1935, built the Army Base for British troops at Harfa; 1936-1937, with J. R. Worcester & Company, designed the McGregor Bridge at Manchester, N. H., and the bridge at Lowell, Mass.; also worked 4 years with the New England Structural Company and 1 year with the New York Central Railroad. 1937 to date with Jackson & Moreland as structural Engineer in charge of design. Refers to *E. H. Cameron, H. A. Gray, W. D. Henderson, B. A. Rich, F. S. Wells.*

JAMES E. LEVINGS, Cambridge, Mass., (b. October 20, 1914, Chicago, Illinois). Graduated from Harvard University in 1936, A.B. Degree. Experience; Summer of 1936, survey for Bowdoin Kent, Iceland expedition; October, 1936 to June, 1938, draftsman and transitman for Gibbs & Hill, Consulting Engineers, New York City; Fall of 1937, Night Student at Brooklyn Polytechnic Institute; 1938 to 1939, student at Harvard Graduate School of Engineering, M.S. Degree in 1939; Spring of 1939, with Maintenance Department of Harvard University, general engineering work, structural design, road layout, supervision of construction, etc.; Summer of 1939, with Howard M. Turner, Consulting Engineer, Boston, Mass.; 1939-1940, assistant in Civil Engineering at Harvard Graduate School of Engineering, further work in engineering studies; July, 1940, to date, with Jackson & Moreland, as structural designer, design and analysis of heavy buildings, machine supports, etc. Refers to *E. H. Cameron, A. Haertlein, W. D. Henderson, B. A. Rich, H. A. Thomas, H. M. Turner.*

held on October 21, 1942, at Northeastern University and will be a joint meeting with the Northeastern Section ot American Society of Civil Engineers. Theodore R. Kendall, Editor of *Contractors and Engineers Monthly,* will be the speaker.

Vice-President Edwards, concluding the items of business of B.S.C.E., introduced Lt. Col. J. W. H. Myrick, President of Boston Post, Society of Military Engineers and guest speaker of the evening, who gave a particularly interesting talk on "Smoke Screens for Protection of Vital Industries and Public Utilities and Their Relation to the Conservation of Fuel."

At the conclusion of the address, the speaker was given a rising vote of thanks.

Adjourned at 9:00 P.M.

EVERETT N. HUTCHINS, *Secretary*

## APPLICATIONS FOR MEMBERSHIP

[September 20, 1942]

The By-Laws provide that the Board of Government shall consider applications for membership with reference to the eligibility of each candidate for admission and shall determine the proper grade of membership to which he is entitled.

The Board must depend largely upon the members of the Society for the information which will enable it to arrive at a just conclusion. Every member is therefore urged to communicate promptly any facts in relation to the personal character or professional reputation and experience of the candidates which will assist the Board in its consideration. Communications relating to applicants are considered by the Board as strictly confidential.

The fact that applicants give the names of certain members as reference does not necessarily mean that such members endorse the candidate.

The Board of Government will not consider applications until the expiration of fifteen (15) days from the date given.

*For Admission*

HERBERT C. CAMERON, Medford, Mass. (b. March 20, 1909, Somerville, Mass.) Graduated from Harvard University in class of 1931, B.S. degree in Civil Engineering. Certificate from Lowell Institute at Mass. Institute of Technology in Air Conditioning and Refrigeration. Experience, 1931-1932, taught Mathematics, Physics and Chemistry in the Public Schools of Medford, substitute teacher. Survey, design and field layout of system of roadways and sidewalks; inspection and supervision of work, Harvard University Maintenance Department; 1932, (Winter) worked as sand-hog in construction of East Boston Vehicular Tunnel, Silas B. Mason, Contractor; 1933-1935, designed all types of heating, ventilating and air conditioning systems including work on central heating supply system. Special design of laboratory setups for biological laboratory, this work for Harvard University Engineering Department; 1935-1936, Turner Construction Company of Boston and New York, worked on office engineering staff, scheduling reinforcing steel, preparation of field details, etc. 1936 to present, Jackson & Moreland, Engineers, Boston, Mass., structural designer on steel and concrete for power plant installations and similar structures. Field inspection of Public Utilities structures as part of appraisal of physical property valuation. Refers to *E. H. Cameron, H. A. Gray, A. Haertlein. W. D. Henderson, B. A. Rich, F. S. Wells.*

WILLIAM FRANCIS CONDON, JR., Watertown, Mass. (b. March 15, 1898,

# OF GENERAL INTEREST

## BOSTON SOCIETY OF CIVIL ENGINEERS SCHOLARSHIP IN MEMORY OF DESMOND FITZGERALD AWARDED TO RICHARD D. SUTLIFF, STUDENT AT NORTHEASTERN UNIVERSITY

Richard D. Sutliff of Gloversville, New York, a senior student, Class of 1942, in the Civil Engineering course at the School of Engineering, Northeastern University, was awarded the Boston Society of Civil Engineers Scholarship in memory of Desmond FitzGerald on April 22, 1942, at a convocation of Students held at Northeastern University. The presentation of the Scholarship of $60 was made by Athole B. Edwards, Vice-President of the Boston Society of Civil Engineers.

# PROCEEDINGS OF THE SOCIETY

## MINUTES OF MEETINGS

### Boston Society of Civil Engineers

SEPTEMBER 23, 1942.—A regular meeting of the Boston Society of Civil Engineers was held this evening in the Grill Room, at the Boston City Club and was called to order by Vice-President Athole B. Edwards. This meeting was a joint meeting with the Society of American Military Engineers, Boston Post. Fifty members and guests were present. Forty-eight persons attended the dinner preceding the meeting.

Vice-President Edwards announced the death of the following members:

President Charles R. Main, who had been a member since December 15, 1909, and who died August 22, 1942.

Edward G. Gushee, who was elected a member December 19, 1877, and who died September 4, 1942.

Leonard E. Schlemm, who was elected a member November 20, 1921, and who died April 29, 1942.

Daniel L. Turner, who was elected a member December 19, 1894, and who died March 12, 1942.

The Secretary reported the election of the following to membership:

Grade of Member: Pere O. Haak, *Paul M. Levenson, John A. McAuliffe, Carl H. Seils.

Grade of Junior: #John E. Bamber.

Vice-President Edwards announced that the Annual Student Night will be

*Transfer from Grade of Junior.
#Transfer from Grade of Student.

## IMPORTANT NOTICE

Enclosed herewith is (1) a fold-insert of (Fig. 3) to be placed in your copy of the OCTOBER, 1942, Journal of the Boston Society of Civil Engineers, between pages 296 and 297 in the report of the B.S.C.E. Committee of Sanitary Section.

(2) The items "Of General Interest", minutes, etc., page 366, please insert these also in your OCTOBER Journal.

EVERETT N. HUTCHINS,
*Secretary*

# JOURNAL

OF THE

# BOSTON SOCIETY

OF

# CIVIL ENGINEERS

## VOLUME XXIX

## 1942

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# ADVERTISEMENTS

*The advertising pages of the JOURNAL aim to acquaint readers with Professional and Contracting Services and Sources of Various Supplies and Materials. You would find it of advantage to be represented here.*

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# BOSTON SOCIETY OF CIVIL ENGINEERS

FOUNDED 1848

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(Term expires March, 1943)

Charles O. Baird
Francis H. Kingsbury
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(Term expires March, 1944)

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