THE VIRTUAL DIFFERENTIAL SETTLEMENT METHOD

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The paper defines the different concepts of "basic," "natural" and "damped" differential settlement and the difficulties involved in their exact determination as well as in evaluation of their influence on the structure.

Complete statical computation of a structure, due to the analytically determined differential settlements (for all points) being, in most cases, impracticable, a method is proposed in which a virtual differential settlement at all supporting points is determined separately, on the basis of a simplified statical scheme and extreme soil conditions. In framed structures, based on separate footings to which the proposed method is confined, several typical situations are considered with their corresponding coefficients: end support and internal support in a two- and multi-bay frame, respectively. Soil properties are represented, instead of by an average value of the modulus of elasticity (or modulus of subgrade reaction), by its maximum value and maximum-minimum ratio:

$$\frac{E_s'{}_{max}}{E_s'{}_{min}} \text{ or } \frac{K_{max}}{K_{min}}$$

permitting allowance for extreme soil conditions.

A formula is derived for a simple case of a two equal span beam with one of its supports settling more than the others under virtual extreme soil conditions. This formula for the resulting differential settlement damped by the rigidity of the structure, can be adapted for more complicated statical schemes and other modes of presentation of soil-elasticity properties. Coefficients of statical conditions for typical cases are tabulated and a numerical example is added to illustrate the procedure.

Introduction

The stability of foundations is conditioned by two main factors:

a) safety against soil failure,
b) safety against structural failure or damage due to soil deformations.

The first factor comprises soil stability, and its analysis entails examination of the limit equilibrium of the latter. The second factor comprises the behavior of the structure and the soil, both separate

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Notation.—The letter symbols used in this paper are defined where they first appear, in the illustrations or in the text, and are arranged alphabetically, for convenience of reference, at the end of the paper.

and joint. The soil may be far below its limit shear stress level, but the deformations of its surface and those of the structure due to these soil movements may result in structural failure or damage.

The final allowable load on the foundation lies within the allowable bearing stress limits in the soil and results in settlements below the danger level. Settlement of the whole structure entails usually no danger as long as it is uniform for all its parts. Connections with other structures (or installations such as pipes) may be damaged, but the structure itself is ordinarily not affected. The same applies for non-uniform but still linear settlement, resulting in tilting of the structure in one piece. This kind of settlement, normally confined to rigid structures, involves an aesthetic aspect, but it can be assumed that a tilt of less than 1/250 can be disregarded (Skempton & MacDonald '56 (1), Bjerrum '61 (2)).

The main structural damage involved in foundation settlement is due to unequal and non-linear settlement of different parts, i.e., the non-linear differential settlement, which forms the subject of the present paper.

Definitions

The factors determining differential settlements can be grouped in three main categories (Terzaghi, '35 (3)):

1. *Basic* (or theoretical) differential settlement is that part of the differential settlement which is due to shape of the stress distribution soil, assuming perfectly homogeneous subgrade and completely flexible superstructure. This part of the settlement can usually be predicted quantitatively if the load conditions and the elastic properties of the soil are more or less simple and known. This definition may be extended to include the influence of the size of the foundation, the effectiveness and character of the load, and all other factors capable of analytical expression based on theoretical or empirical data.

2. *Natural* differential settlement is that which includes, in addition to the basic one, deviations due to inhomogeneity* of the soil, unexpected moisture fluctuations, unexpected load distribution or concentration, and other factors not yielding themselves to analytical treatment. Under this definition the structure is again assumed infinitely flexible.

* "Inhomogeneity" in this case refers to deviations from the mean of the properties of so-called "homogeneous" soil.

3. The *final* differential settlement of the structure is a resultant of the natural settlement and the restraining (damping) effect of the rigidity of the structure, namely the *damped differential settlement.*

An infinitely-flexible structure settles as a direct result of natural settlement of the soil; the latter undergoes complete damping in the case of an infinitely rigid structure, resulting in uniform settlement and in transfer of the loads from the deeper-settled parts of the foundation to their less affected counterparts. Most superstructures which have limited rigidity partially damp the natural settlement and settle due to this damped settlement line. According to the magnitude of this resulting differential settlement the structure either remains undamaged or undergoes varying degrees of damage.

Prediction of Basic Settlement

The settlement of foundation in elastic homogeneous soils can be computed theoretically, and so can the basic differential settlement. For example, the maximum basic differential settlement between center and circumference of a circular flexible foundation plate is 0.36 of the maximum basic settlement at the center, while for a square plate it is 0.50 of the maximum (Timoshenko '51 (4)). In more complicated cases these values may be found from the stress distribution and elastic properties of the soil at different depths and under different parts of the structure. The consolidation theory can be utilized in such cases for regular clay layers. In other soils subgrade moduli or statistical coefficients may be used. The Terzaghi-Peck formulas (1948) (5) permitting extrapolation for foundations of different sizes from the settlement of a 1 sq. ft. test plate are in widespread use. In literature and practice also other methods are to be found which may be of help in analytical determination of settlements and differential settlements on the basis of more or less specified soil properties and load conditions.

Natural Differential Settlement

The theoretical settlement profile obtained by one of the above mentioned methods usually fails to provide a true picture for a variety of reasons. For one thing, even soils defined as fairly homogeneous may differ in their properties within limits of about $\pm$ 50%, and these differences may sometimes have more influence on foundation behavior than the absolute mean (Terzaghi '35 (3)). (Hence the need for data supplied by soil-mechanics laboratories to provide the foundation engineer with a complete picture of the results, instead of confining reports to average values.) Other factors distorting the theoretical settlement distribution are time, the real load distribution, settlements during construction, nearby works, etc. In addition, there is difficulty in isolating the natural settlement, which can only be measured after the building has been completed, when in most cases it is no longer "natural" but affected by the rigidity of the structure. (The only exception is a completely elastic element, such as the flexible bottom of a reservoir.)

Damped Differential Settlement

A rigid structure resists deformations due to soil movement and transfers loads and pressures from the more to the less settling points, thus reducing the potential differential settlement to a restrained or damped one. This damping of the soil movement produces a new load distribution in the structure and new loading conditions on the soil. This interaction of the two factors, controlled by their respective elastic and geometric properties, lends itself to theoretical analysis (Meyerhof '47 (6), Chamecki '56 (7), Yokoo & Yamagata '56 (8)), insofar as these properties are known; but the accuracy of this analysis is limited by that of the available data. The data and settlement values supplied by the soil engineer are averages based on tests with varying degrees of scatter and on theories whose relation to reality is mostly qualitative rather than quantitative. On the other hand, statical analysis of the structures is normally based on simplified assumptions. Bearing in mind that in most computations all secondary elements (such as floors and partitions) are disregarded, discrepancies of up to ± 50% between calculated and actual values should be expected (Skempton & MacDonald (1)). Another source of discrepancies lies in inaccurate assumptions with regard to the elasticity and rigidity of members and joints.

The Proposed Method

In these circumstances, more suitable tools are required for estimating the expected differential settlements and the degree of danger to the structure entailed by them. In the author's opinion, the method outlined below permits prediction of the behavior of the structure with an accuracy commensurate with that of other tools used in foundation design. The proposed method (confined at present to framed structures on separate footings) consists in determining virtual differential settlements at all supporting points of the building. Under "virtual differential settlement" is understood the settlement which corresponds to the worst alternative of soil properties scatter. Under this simple approach accuracy is automatically reduced to a very rough level. Recourse to more accurate tools will not help, since if the settlement under footings undergoes changes within the limits determined by the scatter, any specific assumption with regard to the subgrade modulus will still refer to a single alternative, certainly not the worst. Chamecki's and other methods are based on such assumption in actual disregard of the unlimited number of other (not less real) possible alternatives.

Although for the exact solution an infinity of situations should be considered covering the whole range of subgrade moduli, the proposed method is confined to their extrema, thus reducing the number of situations to minimum.

For this purpose the soil properties should be represented, instead of by an average value of E and K, by their maximum value and maximum-minimum ratio:

$$\varkappa = \frac{E_s'{}_{max}}{E_s'{}_{min}} \tag{1}$$

or

$$\varkappa = \frac{K_{max}}{K_{min}} \tag{2}$$

Now, with the degree of accuracy reduced as noted above, a simplified and time saving statical scheme suffices to find the damping influence of the structure.

Two typical schemes will be considered, each for two positions of the settling support: a two-span and a multi-span structure, and settling of the end or interior support.

Another simplification refers to the possibility of checking each support separately. This is effected by means of the fact that all footings are designed for the same contact pressure or settlement value. Thus, with a single proportion for all supports, the adjoining column load values become superfluous leaving only the footing dimensions in the formulas.

In what follows the damping process is illustrated for a two-span beam on three footings. First, the basic differential settlements ($\Delta\rho^{o}$) and the damped differential settlements ($\Delta\rho$) are calculated on the assumption of fixed subgrade moduli under all footings. In that case the differential settlement may be (theoretically) eliminated altogether by adjusting footing sizes to yield equal settlements. Substituting the maximum-minimum ratio ($\varkappa$) for E (in clays) or for K (in sands) the virtual damped differential settlements are obtained.

It is also assumed that the dimensions of the footings and the character of the subgrade allow neglect of the influence of the load of one footing on the settlement of others.*

Assuming the footings in the first stage as non-settling, we have:

$$R_A^o = P_A + \frac{gl}{2} - \frac{M_B^{(-)}}{l} = R_C^o \qquad (3)$$

$$R_B^o = P_B + gl + \frac{2 \cdot M_B^{(-)}}{l} \qquad (4)$$

where:

R_A^o R_B^o R_C^o — are loads on footings A, B & C, respectively, before settlement,
P_A P_B P_C — loads independent of the differential settlement,
g — load on beam per unit length,
$M_B^{(-)}$ — moment at support B before settlement,
l — span.

Basic subgrade settlement will induce settlements ρ_A^o, ρ_B^o, ρ_C^o in the three supports. Assuming $\rho_B^o > \rho_A^o = \rho_C^o$, the basic differential settlement would be:

$$\Delta \rho_{AB}^o = \rho_B^o - \rho_A^o \qquad (5)$$

The damped differential settlement, resulting exclusively from the basic settlement and structural rigidity will be:

$$\Delta \rho_{AB} = \rho_B - \rho_A = (\rho_B^o - \Delta \rho_B) - (\rho_A^o + \Delta \rho_A) = \Delta \rho_{AB}^o - (\Delta \rho_B + \Delta \rho_A) \qquad (6)$$

where ρ_A and ρ_B are the final settlements of footing A and B, and $\rho_C = \rho_A$; $\Delta\rho_A$ and $\Delta\rho_B$ the changes in footing settlement due to the influence of the rigidity of the structure, as defined later.

This differential settlement reduces the$M_B^{(-)}$ at the middle support by:

* If this assumption does not suit the foundation conditions, more sophisticated calculations have to be made, but the same principles and the same method can be used. Here in this paper, for simplicity of presentation, simplifying assumptions have been made.

$$\Delta M = \frac{3E_C J_C}{l^2} \Delta \rho_{AB} \tag{7}$$

(E_C and J_C being the modulus of elasticity and moment of inertia of the beam), with the following effect on the reactions $\left(\Delta R = \frac{\Delta M}{l}\right)$:

$$R_A = R_A{}^o + \frac{3E_C J_C}{l^3} \Delta \rho_{AB} \tag{8}$$

$$R_B = R_A{}^o - 2\,\frac{3E_C J_C}{l^3} \Delta \rho_{AB} \tag{9}$$

The effect of the change in the reactions on the settlement ($\Delta\rho$) will be determined here with the aid of the subgrade modulus K, calculated from the modulus of elasticity or computed directly on the basis of laboratory (or field) tests, or taken from tables, once the soil has been identified. Although the simplest, it is not the best and the only method available; as pointed out earlier, however, allowance should be made for accuracy considerations before more exact or more refined methods are resorted to.

So, in general,

$$\rho = \frac{\sigma}{K} \tag{10}$$

(ρ — settlement, σ — stress, K — subgrade modulus).

The final differential settlement will be from Eq. (6) and (10):

$$\Delta \rho_{AB} = \rho_B - \rho_A = \left(\rho_B{}^o - \frac{\Delta \sigma_B}{K_B}\right) - \left(\rho_A{}^o + \frac{\Delta \sigma_A}{K_A}\right)$$

$$= \Delta \rho_{AB}{}^o - \left(\frac{\Delta \sigma_B}{K_B} + \frac{\Delta \sigma_A}{K_A}\right) \tag{11}$$

but as: $$\Delta \sigma_A = \frac{R_A - R_A{}^o}{a^2} \qquad \Delta \sigma_B = \frac{R_B - R_B{}^o}{b^2} \tag{12}$$

where:

$\Delta\sigma$ — change in soil stress due to differential settlement

a, b — size of square footings, so, combining Eq. (11) and (12) with (8) and (9):

$$\Delta \rho_{AB} = \Delta \rho_{AB}^{o} - \left(\frac{6 \cdot E_C J_C}{K_B l^3 b^2} + \frac{3E_C J_C}{K_A l^3 a^2} \right) \Delta \rho_{AB} \qquad (13)$$

the following formula is obtained:

$$\Delta \rho_{AB} = \frac{\Delta \rho_{AB}^{o}}{1 + \frac{3E_C \cdot J_C}{l^3} \cdot \frac{2K_A a^2 + K_B b^2}{K_A \cdot K_B \cdot a^2 b^2}} \qquad (14)$$

which permits calculation of the damped differential settlement from the basic one.

With the basic differential settlement $\Delta\rho_{AB}^{o}$ (5) computed by the same method (subgrade coefficient), the complete relationship can be expressed in a single formula (putting $\rho_A^{o} = R_A^{o}/K_A \cdot a^2$ and $\rho_B^{o} = R_B^{o}/K_B \cdot b^2$):

$$\Delta \rho_{AB} = \frac{R_B^{o} K_A a^2 - R_A^{o} K_B b^2}{K_A \cdot K_B a^2 b^2 + \frac{3E_C J_C}{l^3} (2K_A a^2 + K_B b^2)} \qquad (15)$$

which allows the calculation of the damped (basic only!) differential settlement directly.

Direct Determination of the Damped Differential Settlement in Clayey (Elastic) Soils

Assuming elastic behavior of the clay, a simple expression may be used for the subgrade modulus in this soil:

$$K_i = \alpha \frac{E_s'}{B_i} \quad \left(E_s' = \frac{E_s}{1 - \mu^2} \right) \qquad (16)$$

where α is a shape factor (Terzaghi '43 (9); Skempton '51 (10)); B_i is width of the footing. Then:

$$K_A = \alpha \frac{E_s'}{a} \quad K_B = \alpha \frac{E_s'}{b}$$

where from, for equal α and E_s':

$$K_A/K_B = b/a \quad \text{and} \quad \rho_A/\rho_B = \sigma_A \cdot a/\sigma_B \cdot b \qquad (17)$$

Substituting this in the $\Delta\rho_{AB}$ formula (15), and assuming a homogeneous soil (E_s' = const.), we obtain:

$$\Delta\rho_{AB} = \frac{R_B^o\, a - R_A^o\, b}{\alpha\, E_s'\, a\, b + \frac{3E_C\, J_C}{l^3}(2a + b)} \tag{18}$$

Using the well-known rule (Terzaghi-Peck '48 (5)) for adjusting the design pressure of footings so as to yield the same settlements, one can (theoretically, of course) eliminate differential settlements altogether.

For $\rho_B = \rho_A$ we have, from Eq. (17):

$$\frac{\sigma_B}{\sigma_A} = \frac{a}{b} \quad \text{and} \quad \frac{a}{b} = \frac{R_A^o}{R_B^o} \tag{19}$$

so $$R_B^o\, a - A_A^o\, b = 0, \quad \text{i.e.,} \quad \Delta\rho_{AB} \quad \text{vanishes.} \tag{20}$$

Maximum-Minimum Ratio

However, as already mentioned, even homogeneous soils show deviations from their average properties; these may be expressed for clayey soils by the maximum-minimum ratio of the modulus of elasticity:

$$\frac{E_s'{}_{max}}{E_s'{}_{min}} = \varkappa \tag{21}$$

Now, one footing may be supported by soil having E'_{min} and its neighbors by soil with E'_{max}; this extreme case is involved when the maximum natural virtual differential settlement for the discussed structure is to be taken into consideration. Assuming:

$$E_A' = E_C' = E_s'{}_{max} \qquad E_B' = E_s'{}_{min} \tag{22}$$

we have:

$$\frac{E_A'}{E_B'} = \varkappa \quad E_A' = E_B'\,\varkappa \tag{23}$$

the virtual damped differential settlement is then (by (15) and (17)):

$$\Delta\,\rho_{AB} = \frac{R_B^o\,\alpha\, E_A'\, a - R_A^o\,\alpha\, E_B\, b}{\alpha^2\, E_A' E_B'\, a\, b + \frac{3E_C\, J_C}{l^3}(2\alpha\, E_A' a + \alpha\, E_B'\, b)} \tag{24}$$

or

$$\Delta\,\rho_{AB} = \frac{R_B^o\,\varkappa a - R_A^o\, b}{\alpha\, E'_{max}\, a\, b + \frac{3E_C\, J_C}{l^3}(2\varkappa\, a + b)} \tag{25}$$

or, for σ_A and σ_B adjusted according to (19) with the basic differential settlement eliminated:

$$\Delta \rho_{AB} = \frac{R_B{}^{\circ} (\varkappa - 1)}{\alpha E'_{max} b + \frac{3E_C J_C}{l^3} \left(2\varkappa + \frac{b}{a} \right)} \qquad (26)$$

whence $\Delta\rho_{AB}$ can be estimated separately for each footing, with the effect of adjacent footings expressed by the size ratio $\frac{b}{a}$, and span l only.

Direct Damped Differential Settlement in Sand

For sand, using the empirical formula for settlement estimation as proposed by Terzaghi & Peck (5), we have:

$$\rho_i = \rho_o \left(\frac{2B_i}{B_i + 1} \right)^2 \quad (B_i \text{ footing width in feet}) \qquad (27)$$

Hence,

$$K_i = \frac{\sigma}{\rho_i} = \frac{\sigma}{\rho_o} \left(\frac{B_i + 1}{2B_i} \right)^2 = K_o \left(\frac{B_i + 1}{2B_i} \right)^2 = K_o \eta_i \qquad (28)$$

K_i and ρ_i being respectively the modulus of subgrade reaction and settlement of a footing of width B_i (in feet). K_o and ρ_o are, respectively, the modulus of subgrade reaction and the settlement of a 1 sq. ft. test plate. η_i is the size effect factor. The ratio ρ_i/ρ_o is plotted against B_i in Fig. 2. The diagram shows that for small footings only (up to about 6 feet) the size effect is of great importance, while above that limit settlement shows smaller sensitivity to size changes.

The basic differential settlement can be eliminated (for $\varkappa = 1$ and, of course, theoretically only) by designing the footings with adjusted contact pressures (see numerical example). In some cases, assuming that all footings exceed 6 ft., settlement differences due to the size effect may be disregarded, so that the difference would only be due to deviations of K in homogeneous soil.

As already mentioned, the deviations of K are represented by the ratio

$$\varkappa = K_{max}/K_{min} \quad (2)$$

and assuming

$$K_A = K_{max} = \varkappa K_B$$

the formula for the damped differential settlement in sand will be (from (15)):

$$\Delta \rho_{AB} = \frac{R_B{}^o \varkappa a^2 - R_A{}^o b^2}{\varkappa K_B a^2 b^2 + \frac{3E_C J_C}{l^3} (2 \varkappa a^2 + b^2)} \tag{29}$$

Now, with the basic differential settlement due to size effect neglected and footings designed with equal contact pressure $\sigma = \frac{R_B{}^o}{b^2} = \frac{R_B{}^o}{a^2}$ the virtual damped differential settlement will be:

$$\Delta \rho_{AB} = \frac{R_B{}^o (\varkappa - 1)}{K_{max} b^2 + \frac{3E_C J_C}{l^3} \left(2 \varkappa + \frac{b^2}{a^2} \right)} \tag{30}$$

If the size effect is to be taken into consideration in designing the footings, the settlements of $\varkappa = 1$ will be equal.

$$\rho_A = \rho_B = \frac{R_A{}^o}{a^2 K_{oA} \eta_A} = \frac{R_B{}^o}{b^2 K_{oB} \eta_B}$$

from where:

$$R_A{}^o = R_B{}^o \frac{a^2 \eta_A}{b^2 \eta_B} \tag{31}$$

Now, using the formula (15) and assuming $\varkappa = \frac{K_{oA}}{K_{oB}} > 1$ the virtual, damped differential settlement will be:

$$\Delta \rho_{AB} = \frac{R_B{}^o K_{oA} \eta_A a^2 - \left(R_B{}^o \frac{a^2 \eta_A}{b^2 \eta_B} \right) K_{oB} b^2 \eta_B}{K_{oA} \eta_A K_{oB} \eta_B a^2 b^2 + \frac{3E_C J_C}{l^3} (2K_{oA} \eta_A a^2 + K_{oB} \eta_B b^2)} \tag{32}$$

from where:

$$\Delta \rho_{AB} = \frac{R_B{}^o (\varkappa - 1)}{K_{o\,max} \eta_B b^2 + \frac{3E_{oC} J_C}{l^3} \left(2 \varkappa + \frac{(b+1)^2}{(a+1)^2} \right)} \tag{33}$$

By assuming $\eta_A = \eta_B$ (and so neglecting size effect differences at footings larger than 6 feet) the formula (30) is obtained again.

The proposed way would be to design the footings according to equal settlements (with size effect influence), but to neglect this factor in the virtual damped differential settlement formula, using (30) instead of (33), as it is shown in the numerical example. Where pressure-settlement relations are incapable of expression in such simple terms, the formula lends itself to certain refinements, but the idea remains the same.

It can be seen that for a completely flexible structure ($E_C J_C = 0$)

$$\Delta \rho_{AB} = \frac{R_B^o}{\varkappa K_B b^2}(\varkappa - 1) = \rho_B^o \frac{\varkappa - 1}{\varkappa} = \rho_{max} \frac{\varkappa - 1}{\varkappa} \qquad (34)$$

or, for:

$$\varkappa = 2 \quad (K = K_{av} \pm 33\%) \quad \Delta \rho_{AB} = \frac{1}{2} \rho_{max}.$$

$$\varkappa = 3 \quad (K = K_{av} \pm 50\%) \quad \Delta \rho_{AB} = \frac{2}{3} \rho_{max}. \qquad (35)$$

i.e., values close to those given empirically by Terzaghi & Peck (5) for natural differential settlement and in agreement with the order of magnitude obtained for $\Delta\rho/\rho_{max}$ by Skempton & MacDonald (1) in their comprehensive empirical treatment of this subject.

More Complicated Cases

To adapt the formula for more complicated cases (from the structural point of view), certain modification of the coefficients is required, and the expressions (from (26) and from (30)):

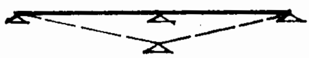
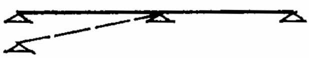
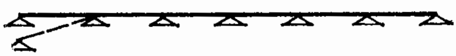
$$\frac{3E_C J_C}{l^3}\left(2\varkappa + \frac{b}{a}\right) \text{ for clays, or } \frac{3E_C J_C}{l^3}\left(2\varkappa + \frac{b^2}{a^2}\right) \text{ for sands}$$

become, in more general terms:

$$\frac{\beta E_C J_C^\Sigma}{l^3}\left(2\varkappa + \frac{b}{a}\right) \text{ or } \frac{\beta E_C J_C^\Sigma}{l^3}\left(\gamma\varkappa + \frac{b^2}{a^2}\right)$$

where B and γ are statical factors of the structure (see Table 1) and J^Σ is the overall moment of inertia of the whole section of a multi-story

TABLE I

Number of bays and position of settling support	β	γ
Two-bay, internal support 	3	2
Multi-bay, internal support 	10.8	1.35
Two-bay, end support 	1.5	2.0
Multi-bay, end support 	1.6	2.3

frame. The two formulae (26) and (30) can thus be written in the general form

$$\Delta \rho_{AB} = \frac{R_B{}^{o}\,(\varkappa - 1)}{\alpha\, E'_{\max}\, b + \dfrac{\beta E_C\, J_C{}^{\Sigma}}{l^3}\left(\gamma\,\varkappa + \dfrac{b}{a}\right)} \quad \text{for clays} \qquad (36)$$

and

$$\Delta \rho_{AB} = \frac{R_B{}^0 (\varkappa - 1)}{K_{max}\, b^2 + \dfrac{\beta E_C\, J_C{}^{\Sigma}}{l^3}\left(\gamma \varkappa + \dfrac{b^2}{a^2}\right)} \quad \text{for sands} \qquad (37)$$

Calculation of the equivalent moment of inertia presents no difficulties in a multi-story and multi-bay frame. The question is, what degree of accuracy is justified in such a calculation. In most cases approximate methods are quite adequate for the order of magnitude of the influence on the differential settlement.

As a general rule, it can be stated that the equivalent moment of inertia is

$$J_C{}^{\Sigma} = 0.6\, \Sigma J_C \quad \text{to} \quad 1.0\, \Sigma J_C \qquad (38)$$

where ΣJ is the sum of J for all beams. But, bearing in mind the increase in the rigidity of the structure due to secondary elements (floors and partitions), it would be justified for this order of accuracy to take the overall moment in view of the fact that measured stresses and deflections are usually about 0.4 to 0.6 of the calculated ones. It thus seems logical to take the overall rigidity of the structure.

Kany (1959) (11) recommends the full ΣJ with the rigidity of partitions or walls allowed for.

The influence of frame supported by the same column in the perpendicular direction may be disregarded. It is assumed that the latter settles in such a way as not to allow transfer of loads between them, causing the virtual worst possible case.* This enables a treatment of the discussed frame as a one-dimensional one.

The footnote from page 204 should also be in mind when discussing the accuracy of the formulae and more sophisticated methods.

Conclusion

Using simplified statical assumptions and extreme soil-elasticity values, a virtual damped differential settlement can be determined separately for each footing of a framed structure foundation. The formulae are flexible enough for adjustment and refinement with a view to a higher degree of accuracy, if required.

This damped differential settlement should be compared with the allowable differential settlement proposed by Terzaghi & Peck (5) or,

* Note: An assumption that the perpendicular frame may also at the same time transfer load to the discussed one, though theoretically possible, seems to be exaggerated compared to the approach presented here.

more recently by Skempton & MacDonald (1), as well as checked by statical calculations as to its influence on the stresses in the structure.

It is suggested here that soil laboratories should provide the necessary data for the foundation engineer, including extreme soil properties rather than their averages.

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Numerical example

The plan and two sections with spans, beam dimensions and typical column loads are given in Fig. 3. The subgrade is medium-loose sand with allowable bearing pressure of about 1.5 ton/sq. ft. The average modulus of subgrade reaction (Terzaghi '55) (12) is:

$$K_o = \overline{K}_{sl} = 100\text{t/cub. ft.} \qquad K_i = K_o \left(\frac{B_i + 1}{2\,B_i} \right)^2 \tag{28}$$

The square footings and their settlements, calculated for equal allowable pressure would be:

$$\begin{array}{lll} 1)\ 5' - 2'' & \rho_1 = 0.51'' & \\ 2)\ 7' - 6'' & \rho_2 = 0.56'' & \\ 3)\ 6' - 7'' & \rho_3 = 0.54'' & \\ 4)\ 8' - 9'' & \rho_4 = 0.58'' & \rho = \dfrac{\sigma}{K} \\ 5)\ 6' - 4'' & \rho_5 = 0.54'' & \\ 6)\ 8' - 2'' & \rho_6 = 0.57'' & \end{array} \tag{10}$$

Footing dimensions, adjusted to equalize all settlements to 0.51″ using Terzaghi's formula (27) will be; $\left(B_i = \sqrt{\dfrac{4\,R_i}{K_o \rho}} - 1 \right)$

1) $5' - 2''$ $(5.17')$ $(\sigma_1 = 1.5$ ton/sq. ft.)
2) $7' - 11''$ $(7.92')$ $(\sigma_2 = 1.36$ ton/sq. ft.)
3) $6' - 9''$ $(6.75')$ $(\sigma_3 = 1.42$ ton/sq. ft.)
4) $9' - 4''$ $(9.33')$ $(\sigma_4 = 1.30$ ton/sq. ft.)
5) $6' - 6''$ $(6.50')$ $(\sigma_5 = 1.43$ ton/sq. ft.)
6) $8' - 8''$ $(8.67')$ $(\sigma_6 = 1.33$ ton/sq. ft.)

Now, assuming that K_o is the average of a series of tests with scatter limits of ± 50%, $\varkappa$ will be:

$$\varkappa = K_{max}/K_{min} = 150\%/50\% = 3 \qquad (2)$$

Each footing should be checked along both the longitudinal and lateral direction of the structure.

Table II contains the data for computation of the virtual differential settlements and the virtual settlements proper using the formula

$$\Delta \rho_{AB} = \frac{R_B^o (\varkappa - 1)}{K_{max}\, b^2 + \frac{\beta E_C J_C^{\Sigma}}{l^3}\left(\gamma \varkappa + \frac{b^2}{a^2}\right)} \qquad (37)$$

TABLE II
COMPUTATION OF VIRTUAL DIFFERENTIAL SETTLEMENTS

(1) Footing No.	(2) R_B^O tons	(3) b ft.	(4) K_{max} t/c.ft.	(5) Direction	(6) $\frac{(b)^2}{(a)^2}$	(7) β	(8) γ	(9) $\frac{E_c I_c}{1^3}$ t/ft.	(10) $\Delta\rho$ inch
1	40	5.17	53	lat.	0.43	1.5	2.0	135	0.35
				long.	0.58	1.6	2.3	203	0.25
2	85	7.92	48	lat.	2.35	3.0	2.0	135	0.32
				long.	0.72	1.6	2.3	203	0.37
3	65	6.75	50	lat.	0.53	1.5	2.0	135	0.43
				long.	1.32	10.8	1.35	203	0.11
4	113	9.33	46	lat.	1.90	3.0	2.0	135	0.38
				long.	1.26	10.8	1.35	203	0.17
5	60	6.50	50	lat.	0.56	1.5	2.0	135	0.42
				long.	1.0	10.8	1.35	203	0.11
6	100	8.67	46	lat.	1.78	3.0	2.0	135	0.36
				long.	1.0	10.8	1.35	203	0.17

For footing 1, the values in the lateral direction are obtained as follows:

$$K_{max} = 150 \left(\frac{5.17 + 1}{2 \times 5.17}\right)^2 = 53 \text{ t/cub. ft.}$$

$$b^2/a^2 = (5.17/7.92)^2 = 0.43 \qquad (28)$$

β and γ are respectively (Table I): 1.5 and 2.0

$$\frac{E_C J_C^{\Sigma}}{l^3} = \frac{2 \times 10^6 \times 1.57 \times 10^6}{20^3 \times 12^3} = 22500 \text{ lb/in} = 135 \text{ ton/ft}$$

where: $E_C = 2{,}000{,}000$ lb/sq. in.

$$J_C^{\Sigma} = \sum_{1.}^{5.} \frac{b\, d^3}{5} = 5 \times \frac{10 \times 25^3}{5} = 157{,}000 \text{ in}^4 \qquad (38)$$

(In longitudinal direction: $J_C^{\Sigma} = 5 \times \dfrac{8 \times 20^3}{5} = 64{,}000 \text{ in}^4$

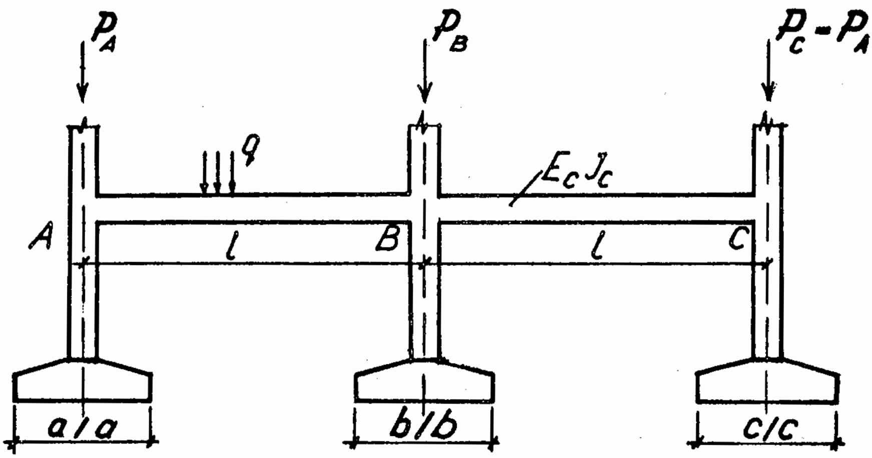

Fig. 1

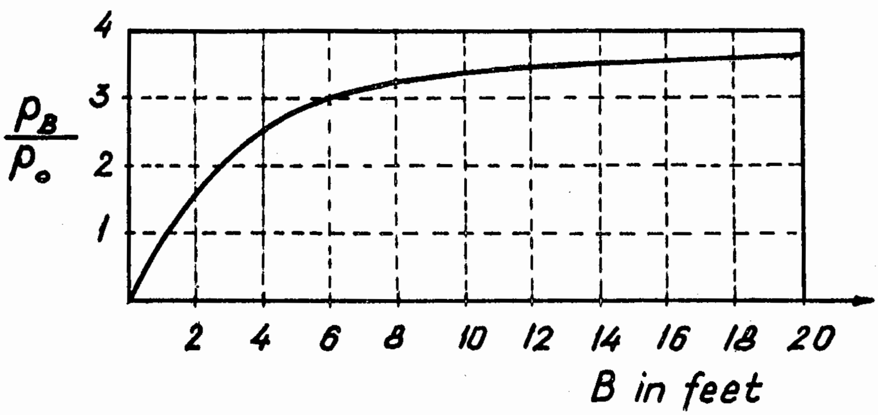

Fig. 2

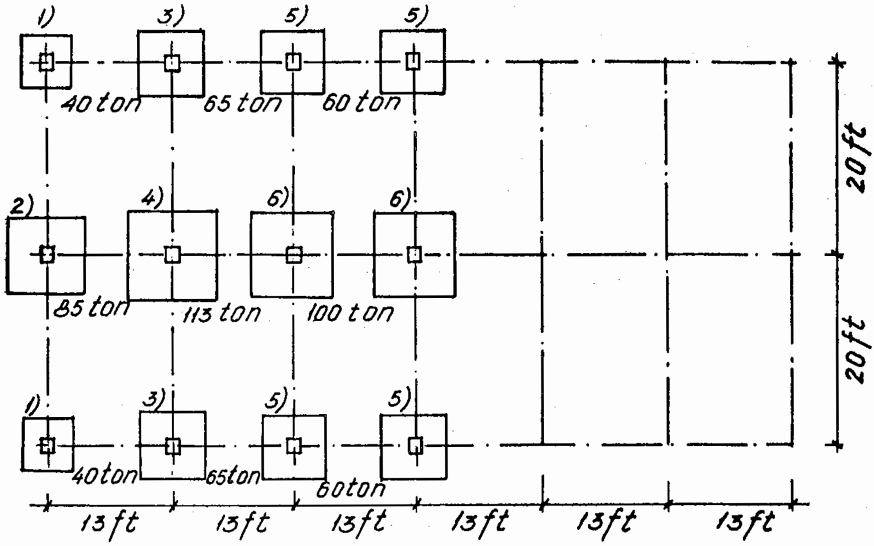
1)
3)
5)
5)
40 ton
65 ton
60 ton
2)
4)
6)
6)
85 ton
113 ton
100 ton
1)
3)
5)
5)
40 ton
65 ton
60 ton
13 ft
13 ft
13 ft
13 ft
13 ft
13 ft
20 ft
20 ft

FIG. 3a

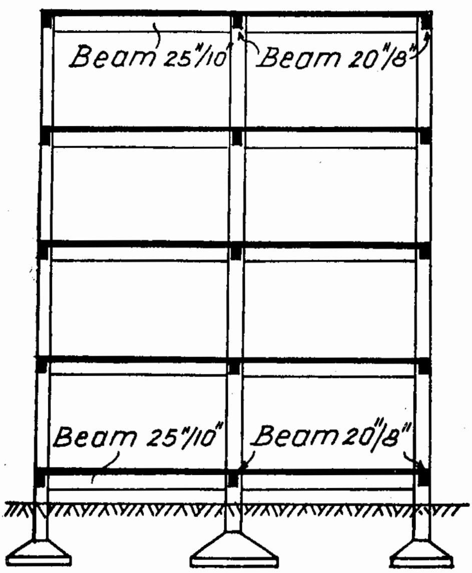
Beam 25"/10"
Beam 20"/8"
Beam 25"/10"
Beam 20"/8"

FIG. 3b

and $\frac{E_C J_C^{\Sigma}}{l^3} = \frac{2 \times 10^6 \times 6,4 \times 10^4}{13^3 \times 12^3} = 33{,}700 \text{ lb/in} = 203 \text{ ton/ft.})$

The virtual differential settlement of footing 1 in the lateral will be:

$$\Delta \rho_{AB} = \frac{40\ (3 - 1)}{53 \times 5.17^2 + 1.5 \times 135\ (2 \times 3 + 0.43)}$$

$$= 0.0296 \text{ ft.} = 0.35 \text{ inch.}$$

It can be seen from comparison (Table II) that although none of them is likely to endanger the structure, it would be reasonable, design-wise, to increase the rigidity of the lateral beams. This would help to keep differential settlements in both directions proportionate to the spans, and to keep the virtual angular distortion within the same order of magnitude.

For more exact values of the differential settlements the formula (33) may be used.

Notation

1.	A, B, C,	— Footings
2.	a, b, c,	— Size of square footings A, B, and C respectively
3.	B_i	— Width of footing (in general)
4.	E_c	— Modulus of elasticity of (concrete) beam
5.	E_S	— Modulus of elasticity of soil
6.	E_S', E_A', E_B'	— $\frac{E_S}{1-\mu^2}$ Modulus of elasticity of soil including Poisson influence
7.	j_c	— Moment of inertia of (concrete) beam
8.	j_c^{Σ}	— Overall moment of inertia of the section of a multistory frame
9.	K_0, K_{0A}, K_{0B}	— Coefficient of subsoil reaction of a 1 sq. ft. test plate
10.	K, K_1, K_A, K_B, K_c	— Coefficient of subsoil reaction
11.	l	— Span
12.	$M_B^{(-)}$	— Moment at support before settlement
13.	ΔM	— Change in moment at support, due to differential settlement
14.	P_A, P_B, P_c,	— Loads, independent of the differential settlement
15.	g	— Load on beam per unit length
16.	$R^o{}_A, R^o{}_B, R^o{}_c$	— Loads on footings A, B, and C, before settlement
17.	R_A, R_B, R_c	— Final loads on footings A, B, and C after settlement
18.	α	— Shape effect factor on footings
19.	β, γ	— Factors of statical conditions
20.	$\varkappa$	— Maximum-minimum ratio of E_S' or K
21.	μ	— Poison coefficient
22.	ρ, ρ_1	— Settlement (in general)
23.	$\rho_A{}^o, \rho_B{}^o, \rho_c{}^o$	— Basic settlements (before damping by structure)
24.	ρ_A, ρ_B, ρ_c	— Final damped settlements
25.	$\Delta\rho_A{}^o{}_B$	— Basic differential settlement between footings A and B
26.	$\Delta\rho_{AB}$	— Damped differential settlement between footings A and B

27. $\Delta\rho_A, \Delta\rho_B$ — Change in settlement due to the influence of the rigidity of the structure
28. ρ_0 — Settlement of a 1 sq. ft. test plate
29. $\sigma, \sigma_A, \sigma_B,$ — Stress in soil
30. $\Delta\sigma, \Delta\sigma_A, \Delta\sigma_B$ — Change in stresses due to differential settlement
31. η_i, η_A, η_B — Size effect factors $\left[\left(\frac{B_i + 1}{2\,B_i}\right)^2\right]$

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