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SOME PRACTICAL IMPLICATIONS OF ELEMENTARY SAFETY ANALYSIS

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A fundamental benefit of the theory of structural reliability is that it puts the understanding of what affects structural safety on a firm basis. It permits one to isolate factors that are widely known to affect safety and to determine new, less obvious, factors. In both cases the factors can be studied quantitatively to determine the situations under which they are significant and, just as important, the circumstances under which they are relatively unimportant. In this paper a number of such factors are identified and known results, determined from previous theoretical studies, are deduced and interpreted in an elementary manner. The intent is to keep the discussion non-theoretical and qualitative in nature, relying upon the reader's intuitive understanding of simple probability notions[3] to justify the conclusions.

The succeeding sections will cover some lessons to be learned from focusing attention on, in turn, the individual components of a structure, a structure subjected to an entire history of loading, complex structure under a known load, and finally a similar structure under an uncertain load.

1. Associate Professor, Massachusetts Institute of Technology
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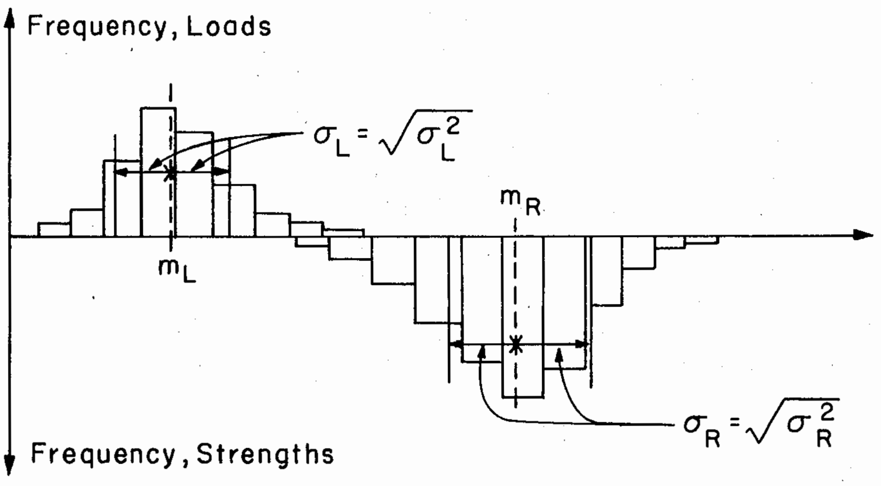
3. Meyer, P. L. "Introductory Probability and Statistical Applications," Addison-Wesley, 1965

LOAD AND STRENGTH VARIATION INFLUENCE

It is widely recognized that the safety of a member is determined not only by the relationship between the typical or central values of material strengths and applied loads, but also by the degree of variability or dispersion demonstrated by both strengths and loads. Owing to the nature of the material and to the conditions under which it is manufactured, relative dispersion in concrete strengths is known, for example, to be larger generally than in steel strengths. This variability influences a member's reliability, i.e., the likelihood that it will perform properly in service. Similarly, the wide variation in earthquake accelerations, storm wind velocities, and numerous other natural loadings makes it unwise to design for their average annual maximum values; their variability too is significant.

But how do these factors influence safety and to what extent? Sketches of histograms of numerous strength and load observations can be converted to similar units, and juxtaposed as shown in Figure 1. Measures of the central or typical values are the average or *mean* load, m_L, and mean strength or resistance, m_R. A measure of the variation in the load is σ^2_L, the *variance*, or moment of inertial of the histogram about its mean:



HISTOGRAMS OF LOAD AND STRENGTH OBSERVATIONS

Figure 1

$$\sigma_L^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - m_L)^2 \qquad 1$$

in which the x_i are the observed values of the load. The square root of this variance, σ_L, called the *standard deviation,* has the same units as the load and its length is plotted on the sketches. Analogous measures of the central value and dispersion of the resistance are also shown, m_R and σ_R.

Clearly a particular member will perform properly only if its particular strength, R, exceeds the particular load, L, to which it is subjected. From the histograms shown one can estimate the proportion of combinations of one strength and one load which will perform satisfactorily (i.e., for which $R > L$), and this proportion or probability is called the *reliability* of the member. Through reliability theory it can be shown (see Appendix II) that this number, P_s, is at least

$$P_s \geq 1 - \frac{K_R \sigma_R^2 + K_L \sigma_L^2}{(m_R - m_L)^2} \qquad 2$$

in which the two constants K_R and K_L depend upon the shapes of the corresponding histograms. This bound may be well below the actual value (which the theory can also determine[4-5]), but the lower bound's simple form facilitates direct interpretation.

This simple relationship can be used to gain an appreciation for the effect of means and dispersions on reliability. It is somewhat more straight forward to discuss one minus the reliability, $P_f = 1 - P_s$, which is called the *probability of failure,* failure to perform satisfactorily with respect to safety or serviceability, depending on the problem. From Eq. 2

4. Freudenthal, A. M., Garrelts, J. M., Shinozuka, M. "The Analysis of Structural Safety", *Journal of the Structural Division,* Proceedings of ASCE, Vol. 92, No. ST1, February, 1966
5. Turkstra, C. J., "A Formulation of Structural Design Decisions," thesis presented to the University of Waterloo, at Waterloo, Canada, 1962, in partial fulfillment of the requirements for the degree of Ph. D. in C. E.

$$P_f \leq \frac{K_R \sigma_R^2 + K_L \sigma_L^2}{(m_R - m_L)^2} \qquad 3$$

Eq. 3 reveals that the probability of failure (as reflected in this upper bound) depends not only on the difference in central values $m_R - m_L$, but also on the dispersions of both resistance and load, and in a symmetrical, additive way.

Effect of Resistance Variation

Letting $V_R = \sigma_R/m_R$ and $V_L = \sigma_L/m_L$, Eq. 3 also can be re-written as

$$P_f \leq \frac{K_L m_L^2 V_L^2 + K_R m_R^2 V_R^2}{(m_R - m_L)^2} \qquad 4$$

In this form it is apparent that, for given central values, m_R and m_L the (upper bound on the) probability of failure increases linearly with the square of the ratio $\sigma_R/m_R = V_R$. This non-dimensional ratio, called the *coefficient of variation,* can be used to compare the variability of different phenomena, such as two materials. Mill tests on the yield stress of mild steel, for example, have shown[6-7] a coefficient of about 7 to 8%, while the same coefficient for concrete compressive stress may vary from 10 to 25% depending upon the quality control exercised.[8]

In addition to inherent material variability, dispersion due to fabrication and other factors must in general be included in the definition of the coefficient of variation of the resistance of a member in place, as will be discussed.

6. Julian, O. G., "Synopsis of the First Progress Report of the Committee on Factors of Safety," *Journal of the Structural Division,* Proceedings of ASCE, Vol. 83, 1957
7. Freudenthal, A. M., "Safety and Probability of Structural Failure," *Journal of the Structural Division,* Proceedings of ASCE, Vol. 121, 1956
8. ACI Standard 214-65, "Recommended Practice for Evaluation of Compression Test Results of Field Concrete," Jan. 1965

Effect of Load Dispersion

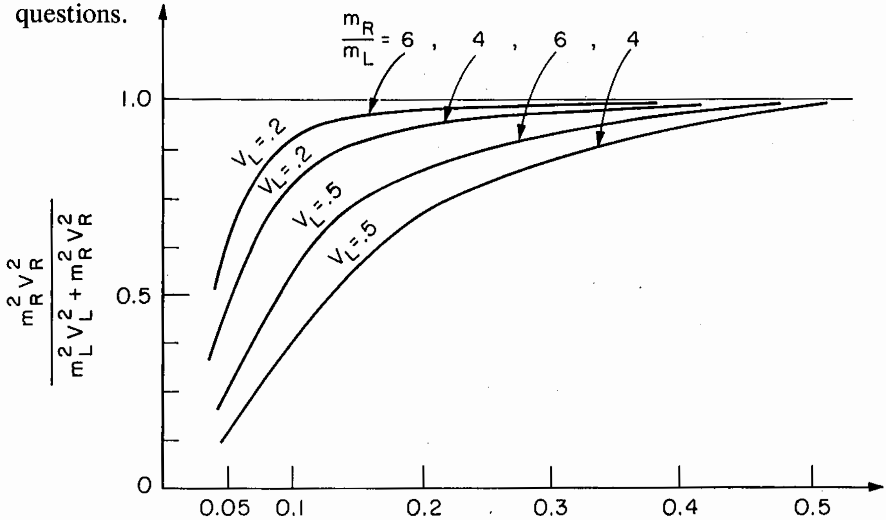
At the same time Eq. 4 shows that the coefficient of variation of the load affects reliability in a parallel way. For given central values, a structure is less safe with respect to wind loads[9] where V_L = 30 to 50% than with respect to dead loads[10-11] where V_L is less than 10%. Most present building codes reflect the influence of uncertainty primarily through load factors of different magnitudes. Although seldom in a quantified manner, nominal working loads and specified strengths may also reflect σ_L and σ_R as well as m_L and m_R. The most recent codes[10] make these relationships more explicit, choosing design loads, for example, proportional to mean (maximum) loads plus a fixed number of standard deviations.

Effect of Resistance Versus Load

It is commonly stated that the uncertainty in the loading on structures is so great that the variability in the strengths of most common construction materials is negligible in comparison. Eq. 4 permits a ready quantitative evaluation of this statement. Notice that the coefficients of variation appear squared emphasizing the difference between V_L and V_R. Representative values of the former might range from 0.1 to a typical value of 0.5 or above, while the latter might range from 0.05 through a typical value of 0.1 or 0.15 to 0.3 or above. The ratio of V^2_L/V^2_R for typical values is of the order of 20, which seems to verify the common statement. Notice, however, that in the expression for (a bound on) the probability of failure (Eq. 4), there appear the products $K_R m^2_R V^2_R$ and $K_L m^2_L V^2_L$. Assuming the shape factors K_R and K_L are about equal, it is the relative values of $m^2_L V^2_L$ and $m^2_R V^2_R$ which should be compared. For a typical value[4] of the "central" safety factor $m_R/m_L = 5$, we find $m^2_L V^2_L/m^2_R V^2_R$ (or σ^2_L/σ^2_R, Fig. 1) about equal to unity for typical values of V_L and V_R. The relative contribution of resistance variation to the upper bound on the probability of failure is shown in Fig. 2 for several cases. The implication is that under typical situations the presently available data fail to justify the common adage that resistance variation has a relatively negligible influence upon the safety of a structure. The larger the value of the central safety

9. Thom, H. C. S., "On Extreme Winds in the United States," ASCE Trans., Vol. 126, 1960
10. C. E. B. Code, "Recommendations for an International Code of Practice for Reinforced Concrete," published by the American Concrete Institute and the Cement and Concrete Association
11. Rosenblueth, E., "Safety in Structural Design," Ch. 19, of a handbook on reinforced concrete design, edited by B. Bresler. To be published.

factor the more important resistance variation becomes. These conclusions, however, are subject to many qualifications, ranging from the lack of sufficient empirical data (both in number and kind), through the assumption that the coefficients of variation are approximately independent of means, to the inadequacy of the theoretical bound in expressing true reliability. Nonetheless the way is clear to begin adequate studies of such questions.



PROPORTION OF RESISTANCE CONTRIBUTION TO TOTAL P_f BOUND VERSUS V_R

Figure 2.

Effect of Dead Versus Live Load

It is possible to go on to investigate the factors which make up the loads and resistances in order to understand better the influences upon structural safety. The total load, for example, is often considered to be the sum of two or more independent loads, say dead load, D, plus live or service load, S. Larger dead load/live load ratios are often stated to be advantageous to safe structures. This conclusion presumably arises from engineers' reflection upon the relative uncertainty in dead and live loads. This uncertainty may be expressed in the variances. If the loads are additive, their means and variances are known to add to be the mean and variance of the total load, L,

$$m_L = m_D + m_S \qquad 5$$

$$\sigma_L^2 = \sigma_D^2 + \sigma_S^2 \qquad 6$$

For given values of the central safety margin, $m_R - m_L$, of the resistance characteristics, m_R and σ_R, and of the shape factors, K_R and K_L, Eq. 4 reveals that the reliability depends upon V^2_L alone. This coefficient can be expanded as follows:

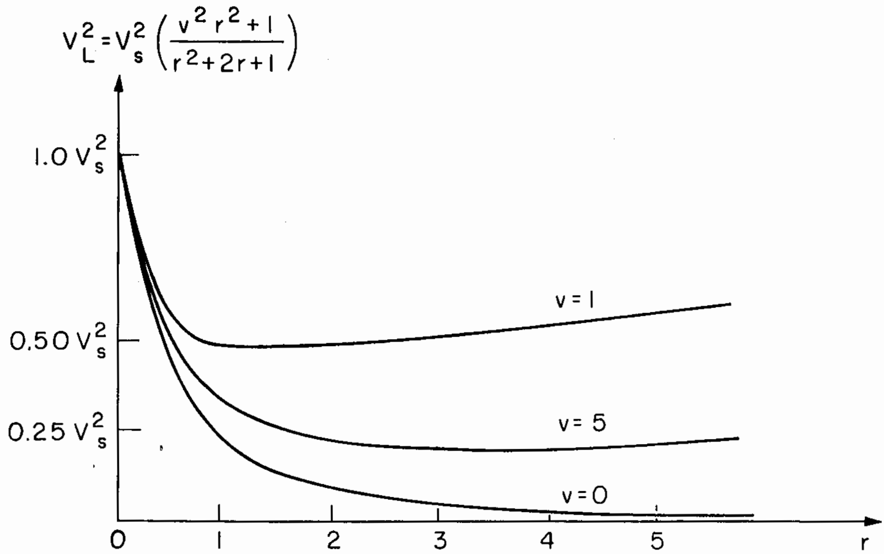
$$V_L^2 = \frac{\sigma_L^2}{m_L^2} = \frac{\sigma_D^2 + \sigma_S^2}{(m_D + m_S)^2} = \frac{V_D^2 m_D^2 + V_S^2 m_S^2}{(m_D + m_S)^2}$$

$$= \frac{V_S^2 m_S^2}{m_S^2} \; \frac{\left(\frac{V_D^2}{V_S^2}\,\frac{m_D^2}{m_S^2} + 1\right)}{\left(\frac{m_D}{m_S} + 1\right)^2} \qquad 7$$

Replacing the ratio of central values of dead load to live load m_D/m_S by r, and the ratio of the corresponding coefficients of variation V_D/V_S by v,

$$V_L^2 = V_S^2 \left(\frac{v^2 r^2 + 1}{r^2 + 2r + 1} \right) \qquad 8$$

In virtually all cases $0 < v < 1$. The above function is plotted for several values of v in Figure 3. Notice that larger dead-load-to-live-load mean ratios, r, are advantageous to safe structures only up to the point where V^2_L reaches its minimum, that is when $r \leqslant 1/v^2$. Beyond this point, V^2_L is a slowly *increasing* function of r. In the range of values of v and r of usual interest however, the intuitive conclusions are analytically justified.



DEPENDENCE OF THE SAFETY UPON THE
DEAD LOAD/LIVE LOAD RATIO

Figure 3

Effect of Resistance Factors

Resistance variation is also not solely a function of one factor. The dispersion in the ultimate moment capacity of nominally similar cross-sections of reinforced concrete beams, for example, depends on the variability of steel yield stress, of concrete compressive stress, and of fabrication. The last factor includes the rolling of the steel bar, the construction of the forms, and the placing of the steel and concrete. A simple approximation (see Appendix III) implies that the variance of the ultimate moment is the sum of the variances of the independent variables multiplied by the squares of corresponding "sensitivity factors." These latter factors indicate the effect of the variable upon the beam's capacity; they are simply the partial derivatives of the ultimate moment with respect to the variables (evaluated at the variables' mean values). Formally

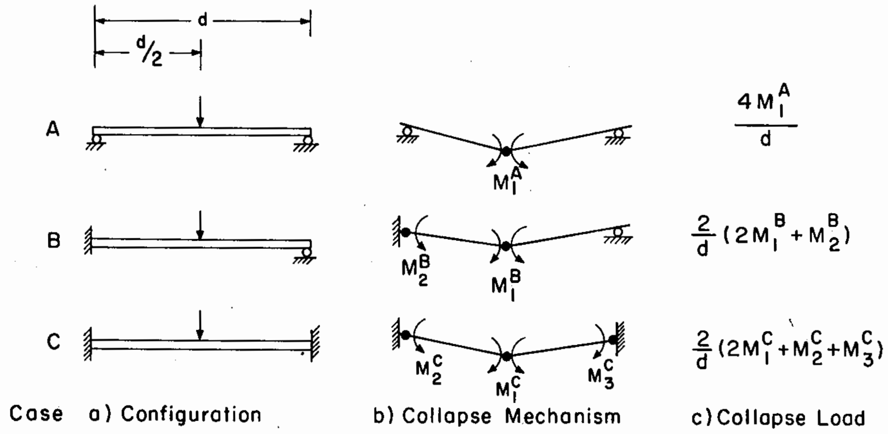
$$\sigma_R^2 \cong \sum \left(\frac{\partial R}{\partial X_i}\right)^2 \sigma_{X_i}^2 \qquad 9$$

in which the X_i's are the variables upon which the cross-sectional resistance depends (steel yield stress, depth of steel, width of beam, etc., in the case of reinforced concrete beam), the $\sigma^2_{X_i}$ are the variances of these variables, and the $\partial R/\partial X_i$ are the partial derivatives of the expression relating R to the X_i's (e.g., the equation for the ultimate moment capacity appearing in the ACI code[12]). Studies involving such approximations and typical values of the mean and variance of the variables can be immediately revealing. Even though concrete compressive strength variation is relatively large, for example, it has a negligible influence on the dispersion of the moment capacity of an under-reinforced cross-section.[13] This is true because the sensitivity factor is very low for this variable, sufficiently so to offset the high variance. Steel yield force and depth of steel contribute the dominant proportions of resistance variation in this case. Variation in latter arises, of course, owing to construction practice and workmanship variability. The shear capacity variance, on the contrary, may depend strongly upon concrete strength variability, depending upon the amount of transverse reinforcement. Such observations, crude as they may be, can have important implications on where an input of effort and expense (such as more strict inspection of certain operations) could most effectively improve structural safety.

Effect of Statical Indeterminacy

Yet another influence upon the safety of a member is thought to be its degree of statical indeterminacy. Reference to "multiple load paths" providing redundancy or to "back up" resistance is often heard in such discussions. Reliability theory can shed light too upon this influence. Consider the three prismatic beams, pictured in Figure 4. By proper choice of the beam sizes the nominal or mean capacities m_R^A , m_R^B and m_R^C can be made equal. Choose the mean moment capacity of a cross-section of Beam B equal to 2/3 that of Beam A, and choose the mean moment capacity of Beam C equal to 1/2 that of a cross-section of Beam A. Assuming perfect elasto-plastic behavior, the mean capacities are then equal (Figure 4):

12. ACI 318-63, "Building Code Requirements for Reinforced Concrete," published by the American Concrete Institute, June, 1963
13. Plum, N. M., "Quality Control of Concrete, Its Rational Basis and Economic Aspects," *Inst. Civ. Eng.*, Proc., Part I, 1953



DETERMINATE VERSUS INDETERMINATE SYSTEMS

Figure 4.

Case A: $$m_R^A = (4/d)m^A \qquad 10$$

Case B: $$m_R^B = (2/d)(2m^B + m^B) = (6/d)m^B = (4/d)m^A \qquad 11$$

Case C: $$m_R^C = (2/d)(2m^C + m^C + m^C) = (8/d)m^C = (4/d)m^A \qquad 12$$

In which m^A, m^B and m^C are the mean values of the moment capacities M_i^A, M_i^B and M_i^C of the cross-section of beams A, B and C, respectively, and d is the distance between the beam supports.

The variances of the beams' capacities may differ considerably, however. Assuming that the coefficients of variation, V^A, V^B, and V^C, of all beams' cross-section capacities are equal, the standard deviations of the cross-sections are

$$\sigma^A = m^A V^A \qquad 13$$

$$\sigma^B = m^B V^B = (2/3)m^A V^A = (2/3)\,\sigma^A \qquad 14$$

$$\sigma^C = m^C V^C = (1/2)m^A V^A = (1/2)\,\sigma^A \qquad 15$$

In which σ^A, σ^B and σ^C are the standard deviations of M_i^A, M_i^B and M_i^C, respectively. The variances of the resistances of the beams follow from elementary probability theory as

$$\sigma_R^2 A = \left(\frac{4}{d}\right)^2 \left(\sigma^A\right)^2 \qquad 16$$

$$\sigma_R^2 B = \left(\frac{2}{d}\right)^2 \left\{ 4\left(\sigma^B\right)^2 + \left(\sigma^B\right)^2 \right\} \qquad 17$$

$$\sigma_R^2 C = \left(\frac{2}{d}\right)^2 \left\{ 4\left(\sigma^C\right)^2 + \left(\sigma^C\right)^2 + \left(\sigma^C\right)^2 \right\} \qquad 18$$

These results depend upon the assumption that the cross-sectional moment capacities are probabilistically independent, a notion which will be returned to shortly. In terms of σ^A or simply σ,

$$\sigma_R^2 A = \frac{16\,\sigma^2}{d^2} \qquad 19$$

$$\sigma_R^2 B = \frac{20}{d^2} \left(\sigma^B\right)^2 = \frac{40}{3d^2}\,\sigma^2 = \frac{13.3\,\sigma^2}{d^2} \qquad 20$$

$$\sigma_R^2 C = \frac{24}{d^2} \left(\sigma^C\right)^2 = \frac{12}{d^2}\,\sigma^2 \qquad 21$$

Under these conditions, although mean member resistances are equal, the variability in the resistance decreases, while reliability increases, as the degree of indeterminacy grows. Reliability theory here supports and quantifies the engineer's intuition about ductile, indeterminate structures. (Brittle structures do not share this advantage.)[14]

14. Shinozuka, M., "On Fatigue Failure of a Multiple-Load-Path Redundant Structures," Vol. 2, Proc. of the First Intern. Conf. on Fracture, Sendar, Japan, 1965

A critical assumption in the previous argument was that of probabilistic independence among the cross-section capacities within a beam. Such independence implies the assumption of a lack of correlation or coherence among these capacities compared to the capacities of all beams. In fact, owing to their common background (same batch of steel, same rolling and cooling experience, etc.), some degree of probabilistic dependence undoubtedly exists among the cross-section capacities within a beam. If the capacity of one cross-section in a beam is above the average among all such beams, it is very likely that other cross-sections of the same beam are also above average. In other words, if the capacity of one cross-section in a beam were tested and found to be a particular value, say 10% more (or less) than the population average, another cross-section in the same beam will most likely have a capacity very near that same value, rather than continuing to be about equally likely to be either higher or lower than the population average.

Such correlation is subject to experimental estimation. Its effect on the previous results can be large.[15–16] If as a limit, the correlation is perfect, it implies that all cross-sectional capacities (although still subject to variation from beam to beam) are equal within a beam. In Case B, this perfect dependence would imply $M_1^B = M_2^B$. In this case

$$\sigma_R^2 B = \left(\frac{6}{d}\right)^2 (\sigma^B)^2 = \frac{36}{d^2}\left(\frac{4}{9}\sigma^2\right) = \frac{16}{d^2}\sigma^2 \qquad 22$$

While in Case C

$$\sigma_R^2 C = \left(\frac{8}{d}\right)^2 (\sigma^C)^2 = \frac{64}{d^2}\left(\frac{1}{4}\sigma^2\right) = \frac{16}{d^2}\sigma^2 \qquad 23$$

In short, if this dependence or correlation is very nearly perfect, which intuition and initial evidence[15] suggests is the case, the variability of an indeterminate beam is no less than that of a determinate one. Consequently it is no more safe. It is *unconservative* to ignore this de-

15. Cornell, C. A., "Bounds on the Reliability of Structural Systems," *Journal of the Structural Division,* Proceedings of ASCE, Vol. 93, No. ST1, February, 1967
16. Tichy M. and Vorlicek M., *Safety of Reinforced Concrete Framed Structures,* Proceedings of the International Symposium on Flexural Mechanics of Reinforced Concrete, Miami, Fla., November, 1964, ASCE, 1965

pendence in a safety study of redundant structures. Reliability theory thus may reveal that there exist unsuspected influences on structural safety. Probabilistic independence versus dependence is one such influence that appears throughout more thorough reliability studies.[15-16-17]

LENGTH-OF-LIFE INFLUENCE ON STRUCTURAL SAFETY

A structure which must serve for a longer time should be designed more conservatively. This common axiom too is easily investigated through some elementary ideas of structural reliability analysis. Given the reliability, P_S, of a member or structure subjected to a single load, the reliability of the same structure subjected to a sequence of such loads can be estimated as follows.

Elementary probability theory states that the probability of two independent events occurring is the product of their probabilities. So the probability of getting two successive heads on two flips of a well-balanced coin is (½) (½) or ¼. If the events are not independent, but dependent or related in some way, the second factor must be replaced by the *conditional probability* of the second event given that the first has occurred. So the probability that the result of the throw of the die is both odd *and* less than four is (½) (⅔)= ⅓. The first factor is the probability of an odd number, 1, 3, or 5. The second factor is the conditional probability of a number less than four *given* that the outcome was odd, 1 or 3. (The answer is *not* (½) (½), the probability of an odd number times the probability of a number less than four.) So, to determine the probability that a structure survives two load applications, P_{S_2}, we must multiply the probability of a survival on the first load application, P_{S_2}, times the conditional probability of survival on the second given survival on the first, $P[s_2 \mid s_1]$. (Read $P[A \mid B]$ as the probability of event A given that event B occurred.)

$$P_{s_2} = P_s P[s_2 \mid s_1] \qquad 24$$

17. Ang, A. H. S., and Amin, M., "Studies of Probabilistic Safety Analysis of Structural Systems", University of Illinois Civil Engg. Studies, Struct. Res. Series No. 320, 1967

(If the successive loads are taken to be the annual maximum loads, the number of loads can be equated to years of service.)

The value of this conditional probability may range from a number as low as P_s to one as high as unity.[5-15] If the successive loads are independent, the conditional probability *may* be about P_s. Then

$$P_{s_2} \cong P_s P_s = (1 - P_f)(1 - P_f) \cong 1 - 2P_f \qquad 25$$

For n loads the result becomes

$$P_{s_n} \cong 1 - n\, P_f \qquad 26$$

In this case the reliability of the structure (now defined as the likelihood that it will perform satisfactorily throughout its lifetime) decays almost linearly with the number of anticipated loads. This conclusion is a quantitative verification of the statement opening this section. It now might be re-stated: "if a structure must serve satisfactorily for n years, it should be designed with a probability of unsatisfactory performance under one annual maximum load, i.e., with a P_f, equal to about one-half that of a similar structure which need serve only for n/2 years."

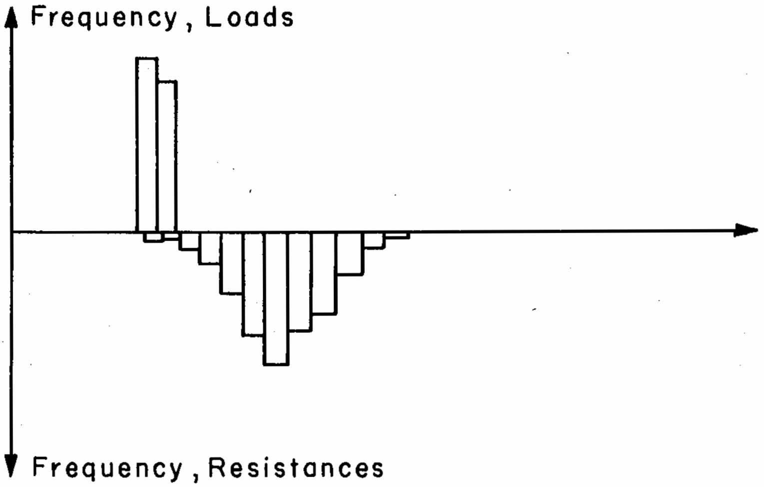
The situation may not, however, dictate that the conditional reliability, $P[s_2 \mid s_1]$ is equal to P_s. For example, if there is virtually no variation in the load, but resistance variation is large, then knowledge that the structure performed adequately under the first load almost guarantees that a low resistance does not exist and hence that in following years the performance will be equally satisfactory. The first load has in essence performed a proof test. This case is illustrated in Figure 5.

In this case

$$P[s_2 \mid s_1] \cong 1 \text{ and}$$
$$P_{s_2} \cong P_{s_1} = 1 - P_f \qquad 27$$

and more generally

$$P_{s_n} \cong 1 - P_f \qquad 28$$



SMALL LOAD VARIABILITY SITUATION $V_L \ll V_R$

Figure 5.

Such a situation occurs in the static design of a dam. The maximum load each year may be always very nearly the capacity of the dam, while the resistance involves significant uncertainty in material properties, abutments strengths, etc. Also, if the significant uncertainty in a structure's resistance lies not in material variability, but in possible design or workmanship errors, blunders, or omissions, the first major load (even if it is only an average one) will be likely to cause the failure *if* it is going to occur. Many construction failures support this observation. In these situations the reliability is not as length-of-life sensitive. (Time-dependent or deteriorating strength is not under consideration here, although this, too, can be treated.[4-15-18])

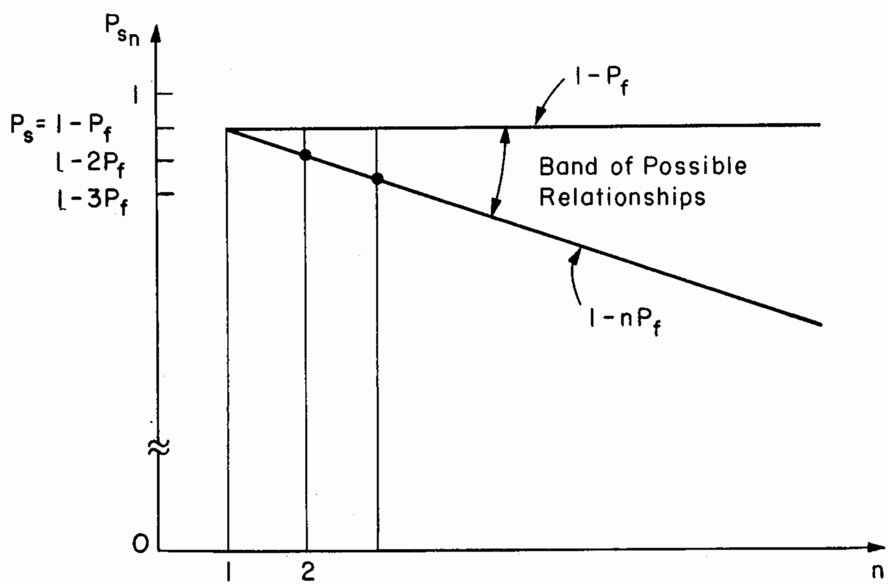
18. Leve, H. L., "Reliability Framework for Structural Design," *Journal of the Structural Division,* Proceedings of ASCE, Vol 89, No. ST1, August, 1963

Quite another situation may lead to the conclusion that $P[s_2 \mid s_1] \simeq 1$. Here, the relatively unfamiliar notion of probabilistic dependence is again the critical factor. If the successive loads are highly correlated, as say dead loads are, then knowledge that in the first year the maximum load didn't exceed the capacity implies that future loads (which are not necessarily well known, but, owing to dependence, will be of very nearly the same magnitude as the first) will also fail to exceed the capacity. The conclusion is, then, the same as Eq. 28.

In general, for n loads the likelihood of satisfactory performance lies between

$$1 - nP_f \leq P_{s_n} \leq 1 - P_f \qquad 29$$

as illustrated in Figure 6. The lower bound[4-15] $1 - nP_f$ may estimate well the common case where load variability, σ_L, exceeds significantly resistance variability, and when successive load values may be treated as independent.

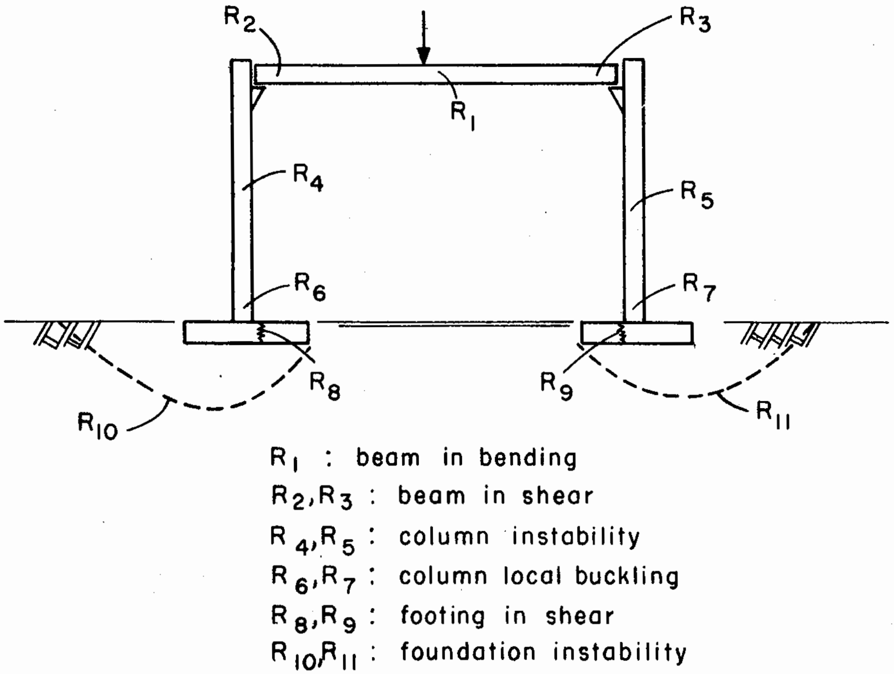


INFLUENCE OF TIME ON STRUCTURAL SAFETY

Figure 6.

STRUCTURAL SYSTEM RELIABILITY UNDER GIVEN LOAD

When structures become more complicated than single members, their safety may be jeopardized by failure of any of several potential modes of failure. A determinate truss will fail if any bar fails. A building may collapse owing to one of several potential weaknesses in its frame, or due to a shear failure of a footing, or because of a foundation stability problem (Figure 7). The analogy is sometimes made to a chain of many links in order to argue that the structural system is no stronger than its weakest mode. Hence the number of modes and the safety of *each* member or mode must influence the total system's reliability.



SCHEMATIC COLLECTION OF POTENTIAL MODES OF COLLAPSE

Figure 7.

These effects are best studied in isolation of loading variation, the influence of which will be discussed in the next section. Assume then that the load is given exactly, or that it demonstrates relatively little dispersion about its known central value, or that for artificial reasons, such as a legally specified maximum load, interest lies only in the safety of the structure under a given load.

By an argument parallel to that in the previous section it is clear that Ps, the likelihood of survival of two modes of the system, is the product of P_1s, the reliability of the first mode or member, times $P[{}_2s \mid {}_1s]$ the conditional reliability of the second mode *given* survival of the first.

$$P_s = P_{1^s} \; P\left[{}_2s \mid {}_1s\right] \tag{30}$$

Since the load is fixed, if the capacities of the first and second modes are probabilistically independent, survival of the first contains no information about the safety of the second. Thus $P[{}_2s \mid {}_1s] = P_2s$, the reliability of the second mode. Then

$$P_s = P_{1^s} P_{2^s} = (1 - P_{2^f})(1 - P_{2^f}) \cong 1 - (P_{1^f} + P_{2^f}) \tag{31}$$

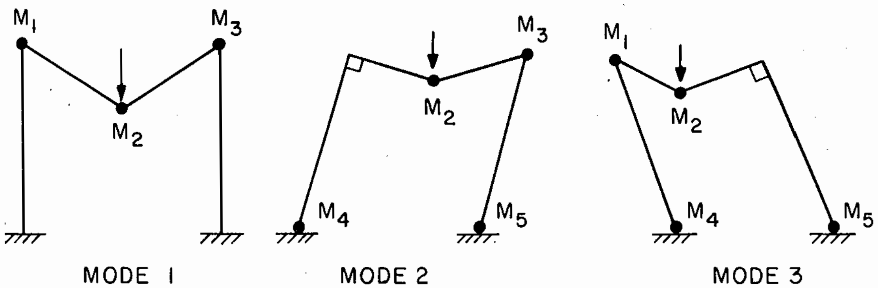
in which P_1f and P_2f are, respectively, the probabilities that the capacities of modes 1 and 2 are insufficient. Under these circumstances the system probability of failure, P_f, is approximately equal to the sum of the m modes' probabilities of failure.[4-15]

$$P_f \cong \sum_{i=1}^{m} P_{i^f} \tag{32}$$

The conclusion is that each mode "contributes" to the system failure likelihood in an almost additive way. Under these circumstances, more complex systems can be made as safe as simple systems only if one increases the individual modes' reliabilities in order to maintain the *sum* of their failure probabilities at the desired value.

In fact, the assumption of independence of the capacities of the modes may not be realistic in many cases. It is true that in the "structure" in Figure 7 the foundation instability modes, R_{10} and R_{11}, which depend on soil properties, may be unrelated to the strength of the beam in bending, R_1. But, the strengths of the columns and the beam may very well be correlated with one another because of common production and fabrication histories.

More directly, in some structures modal resistances may be probabilistically dependent because they are functionally related to the same cross-section capacities.[15-16] Consider, for example, the three simple collapse modes of the portal frame in Figure 8. The first and second modes have resistances which could be treated as the indeterminate beams in Figure 4 were. But now both modal resistances depend in part upon the same cross-section capacities, M_2 and M_3. Given that the capacity of one mode is higher than average, the capacity of the other is very likely to be similarly located with respect to the population of all such modes, simply because M_2 and M_3 are probably greater than average values. Additionally, as discussed in an earlier section and above, the various cross-section capacities are undoubtedly correlated. Owing to their occurrence in the same member or in neighboring members, they are likely to have undergone similar previous histories.



COLLAPSE MODES SHARING COMMON HINGES

Figure 8.

If, as a limiting case, this dependence is perfect, the fact that the mode most likely to fail survived will imply that all others survived also. Then all the conditional survival probabilities of the remaining modes are unity. For two modes,

$$P_s = P_{1^s} \, P\left[{}_2s \mid {}_1s\right] = P_{1^s} \qquad 33$$

in which P_{1^s} is the smaller of the P_{i^s}. Or, in general,[15]

$$P_s = \min_i (P_{i^s}) = 1 - \max_i (P_{i^f}) \qquad 34$$

or

$$P_f = \max_i (P_{i^f}) \qquad 35$$

In reality the dependence among some modal resistances will be strong and among others it will be weak. In some cases one might obtain a good estimate of the system reliability by looking first at appropriate groupings or sub-systems of highly dependent modes. In Figure 7 one might group together the modal resistances one through seven, namely those associated with the steel frame. The probability of failure of such a subsystem would be very nearly the maximum of all the modal probabilities within the subsystem, one of the column instability modes, say, in Figure 7. There may be, however, little correlation between resistances in different subsystems. Hence, the probability of failure of the system would be approximately equal to the sum of the subsystem probabilities of failure. In Figure 7, the sub-systems in addition to the steel frame might be the concrete footings, R_8 and R_9, and the soil instability modes, R_{10} and R_{11}. Then the system reliability under a given load would be approximately

$$P_s \cong 1 - P_{4^f} - P_{8^f} - P_{10^f} \qquad 36$$

In general, about any system or subsystem it can be stated that,

$$1 - \sum_{i=1}^{m} P_{f_i} \leq P_s \leq 1 - \max\left(P_{f_i}\right) \qquad 37$$

Again probabilistic dependence plays a key role in determining the safety of structures. In this case, however, a conservative lower bound is found by ignoring it.

STRUCTURAL RELIABILITY UNDER AN UNCERTAIN LOAD

When the load is not known to be a given value but displays marked variation about its central value, the results of the previous section are altered. In fact, they are simplified. For in this case, even if the capacities of the various modes are independent, probabilistic dependence is set up among the survival *events* by fact that each mode is subjected to the same load, or same loading environment. This *environmental* dependence[15-18-19] causes the conditional probabilities such as $P[{}_2s \mid {}_1s]$ to be very close to unity, when Eq. 35 again holds:

$$P_f \cong \max\left(P_{f_i}\right) \qquad 35$$

This dependence is best understood by considering the limiting case when the resistances are all fixed in value, i.e., lacking any variability. Then, if it is known the weakest mode survived, it is known that the load was less than *all* the resistances, implying all resistances survived. More generally, this condition holds in approximation when the variances of the resistances are small compared to those of the load. This environmental dependence operates, to a greater or lesser extent, any time that the load displays some variation. The degree of its effect is not well understood except in the limiting case mentioned above where the variation in the resistances (or at least in those resistances with the larger $P_i f$) is negligible compared to that in the load.

Nonetheless, the multitude of factors causing dependence among the modal survival events (namely dependence among cross-section re-

19. Moses, F. and Kinser, D. E., "Analysis of Structural Reliability", private communications

sistances, dependence among modal resistances, and environmental dependence) suggests that in reality the safety of complex, multiple-mode-of-failure systems is closer to $1 - \max(P_i f)$, the reliability of its most unreliable mode, than to $1 - \Sigma P_i f$, a cumulative composite of the many modes of failure. The immediate design implication is that it does not create a significantly safer structure if one designs an already non-critical mode to be even more conservative. The design effort should go into the most unreliable mode, but only until it is safer than the next least reliable mode of failure, when any additional expense should go into the latter. Readers familiar with critical path scheduling will note the analogy[20] of this procedure with the one used in improving a critical path schedule.

Combining this conclusion with the result judged to be "most common" in the section on sequences of loads, it is concluded that the reliability of a typical structural system over a period of n years is

$$P_{s_n} \cong 1 - n \max\left(P_{i^f}\right) \qquad 38$$

in which the $P_i f$ are the probabilities of failure of the various modes with respect to a single (annual maximum) load.

It is important to realize that these conclusions extend beyond the study of total collapse. A code (such as ACI 63[12]) might choose to define as failure of the structural system the yielding or crushing of any cross-section of a frame. In this case, each potential yield region becomes a possible mode of failure. Assuming consistent levels of reliability are being sought from building to building, the fact that the code does not require higher safety factors as the number of such cross-sections increases can be justified only if high dependence among survival events exists (i.e., only if P_S equals $1 - \max(P_i f)$ not $1 - \Sigma P_i f$.) Unsatisfactory performance with respect to excessive cracking or excessive lateral sway may be defined as a "serviceability failure" and studied in the same way.[21] In these situations, the determination of the probability of no failure may not be sufficient, because some such failures are

20. Byers, William G., private communications.

21. Robertson, L. E., and Chen, P. W., "Glass Design and Building Code Implications of Recent Wind Load Research for Tall Buildings," Building Research Institute Fall Conference, Nov. 15-17, 1966

tolerable. It becomes important to estimate the number of such failures, for example the number of windows which might crack under high wind loads. If dependence is in fact high, the failure of one mode (window) might imply that several or many more have failed too. (Physically, the same extreme gust, if it occurs, is likely to break several of the weaker windows not just the weakest. These failure events, having been caused by the same gust, are not independent.)

SUMMARY AND CONCLUSIONS

While the effort is not exhaustive (multiple kinds of loads, for example, are not considered), a number of the commonly stated factors affecting structural safety have been reviewed through elementary reliability analysis. These include dispersion in strength and load, the relative influence of resistance and load uncertainty, the dead load/live load ratio, statical indeterminacy, the length of the structure's lifetime, and the complexity of the system. In most cases quantitative verifications have been demonstrated, along with indications as to the degree of significance and conditions for validity of these notions. In still other cases some less familiar factors, most notably probabilistic dependence, have been found critical. The relative variability of load and resistance variation has been seen to be a significant factor in determining the degree to which the length of lifetime and the complexity of the structure affect its safety. The likelihood of a structure's failure may depend most strongly on its most unreliable potential mode of failure rather than on the total of the various modes' unreliabilities.

APPENDIX I — NOTATION

The following symbols are used in this paper:

b = positive constant

D = dead load

d = distance between beam supports

$K_{\bar{R}}$, $K_{\bar{L}}$ = constants depending on the shape of the probability distribution of R and L, respectively

L = load

M_1^A, M_i^B, M_i^C = moment capacities of the cross-sections of beams of A, B, and C respectively

M_i = moment capacity of cross-section i

m^A, m^B, m^C = mean value of M_1^A, M_i^B and M_i^C, respectively

$m_{\bar{D}}$, m_L, m_R, m_S = mean values of D, L, R and S, respectively

m_{R^A}, m_{R^B}, m_{R^C} = nominal or mean capacity of beams of A, B and C respectively

m_{X_i} = mean value of X_i

n = number of loads, or time units (years, say)

P_f = probability of failure

P_{if} = probability of failure of the i^{th} mode, or member

P_{is} = reliability of the i^{th} mode, or member

P_S = reliability (probability of survival)

P_{S_i} = probability that the structure survives i load applications

$P[s_2 \mid s_1]$ = conditional reliability at the second load application, given survival on the first

$P[{}_2s \mid {}_1s]$ = conditional reliability of mode 2 given survival of mode 1

$Q = R - L$ = safety margin

R = capacity or resistance

r = ratio m_D/m_S

R^A, R^B, R^C = capacity of beams A, B and C respectively

S = live load or service load

V^A, V^B, V^C = coefficient of variation of $M_1{}^A$, $M_i{}^B$ and $M_i{}^C$, respectively

V_D, V_L, V_R, V_X = coefficient of variation of D, L, R and S, respectively

x_i = observed value of the load

X_1, X_2, . . . = Variables on which the cross-sectional resistance depends

$\sigma = \sigma^A$

σ^A, σ^B, σ^C = standard deviation of MA, MB and MC, respectively

σ_D, σ_L, σ_R, σ_S = standard deviation of D, L, R and S, respectively

$\sigma_R{}^A$, $\sigma_R{}^B$, $\sigma_R{}^C$ = standard devaiation of the capacity of beams A, B, and C respectively.

APPENDIX II

It will be shown in this section that, for positive load L and resistance R[22]

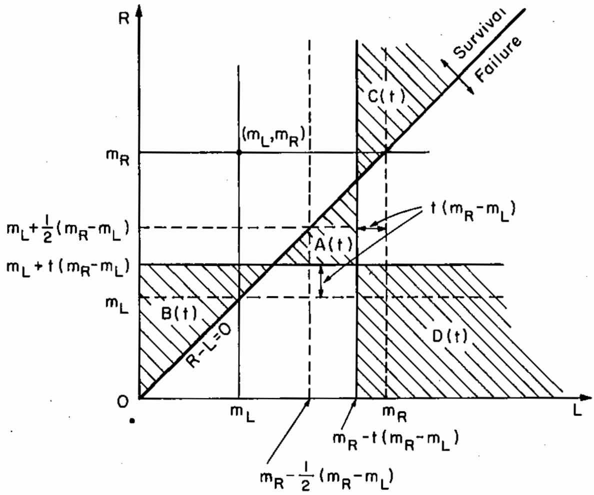
$$P_f \leq \frac{K_R \sigma_R^2 + K_L \sigma_L^2}{(m_R - m_L)^2} \qquad 3$$

in which K_R and K_L are factors, which depend on the shape of the probability density function of R and L, resp.

Proof:

(1) From the representation of the two-dimensional sample space (Figure 9), for any t ≤ ½

22. Vanmarcke, E. H., "Reliability in the Design of Structures," thesis presented to the faculty of the University of Delaware in partial fulfillment of the degree of Master of Science in Civil Engineering



SAMPLE SPACE OF LOAD AND RESISTANCE

Figure 9.

$$P_f = P\left[R - L \leq 0\right] = P\left[L \geq m_R - t(m_R - m_L)\right] + P\left[R \leq m_L + t(m_R - m_L)\right] - \varepsilon(t) \qquad 39$$

in which

$$\varepsilon(t) = P\left[B(t)\right] + P\left[C(t)\right] + P\left[D(t)\right] - P\left[A(t)\right] \qquad 40$$

It follows that

$$P_f \leq P\left[L \geq m_R - t(m_R - m_L)\right] + P\left[R \leq m_L + t(m_R - m_L)\right]$$

if $\varepsilon(t) \geq 0$ 41

Note that for $t = ½, \varepsilon$ (t) is strictly positive, since the only negative term in (40) vanishes.

Both terms on the right hand side of (41) can be approximated by conservative inequalities of the Chebychev type.

(2) For any random variable X with mean m_X and standard deviation σ_X, and for any constant $a \leq m_X$

$$P\left[X \leq a\right] = P\left[m_X - X \geq \frac{(m_X - a)\,\sigma_X}{\sigma_X}\right]$$
42
$$= c_X\, P\left[\left|m_X - X\right| \geq \frac{(m_X - a)\,\sigma_X}{\sigma_X}\right]$$

in which $c_X \leq 1$. If the probability density function is symmetrical about m_X, then $c_X = ½$.

Also, for any such random variable X, having a finite variance and for every $b > 0$, the Chebychev inequality holds.[23-24-5]

$$P\left[\left|m_X - X\right| \geq b\,\sigma_X\right] \leq b^{-2} \qquad 43$$

A less conservative approximation, known as the Gauss inequality, applies if the distribution of X is known to be unimodal with mode $\mathcal{M}_X$. Then[23-24]

$$P\left[\left|m_X - X\right| \geq b\,\sigma_X\right] \leq \frac{4}{9}\,\frac{(1 + \lambda_X^{\,2})}{\left(1 - \frac{|\lambda_X|}{b}\right)^2}\,b^{-2} \qquad 44$$

provided $b > |\lambda_X|$, where $\lambda_X = \dfrac{m_X - \mathcal{M}_X}{\sigma_X}$ 45

In particular, when mean and mode coincide (e.g., when X is symmetrically distributed), then $\lambda_X = 0$, and

$$P\left[\left|m_X - X\right| \geq b\,\sigma_X\right] \leq \frac{4}{9}\,b^{-2} \qquad 46$$

All these bounds can be put in the form

$$P\left[\left|m_X - X\right| \geq b\,\sigma_X\right] \leq k_X\,b^{-2} \qquad 47$$

23. Mood, A. M., "Introduction to the Theory of Statistics," McGraw Hill, 1950
24. Wadsworth and Bryan, "Introduction to Probability and Random Variables," McGraw Hill, 1960

in which $k_X = 1$ corresponds to the Chebychev inequality, and

$$k_X = \frac{4}{9} \frac{(1 + \lambda_X^2)}{(1 - \frac{|\lambda_X|}{b})^2} \qquad 48$$

corresponds to the Gauss inequality.

Finally, introducing (47) into (42), with $b = \frac{m_X - a}{\sigma_X}$ yields

$$P\left[X \leq a\right] \leq \frac{c_X k_X \sigma_X^2}{(m_X - a)^2} \qquad 49$$

In an analogous way it can be shown that, for any random variable X with mean m_X and finite variance σ_X^2, and for any $a' > m_X$

$$P\left[X \geq a'\right] \leq \frac{c_X k_X \sigma_X^2}{(a' - m_X)^2} \qquad 50$$

(3) From (49) and (50), in which a is substituted by $m_L + t(m_R - m_L)$, a′ by $m_R - t(m_R - m_L)$ and X by R and L, resp.,

$$P\left[L \geq m_R - t(m_R - m_L)\right] + P\left[R \leq m_L + t(m_R - m_L)\right]$$

$$\leq \frac{1}{(1 - t)^2} \frac{c_L k_L \sigma_L^2 + c_R k_R \sigma_R^2}{(m_R - m_L)^2} = \frac{K_L \sigma_L^2 + K_R \sigma_R^2}{(m_R - m_L)^2} \qquad 51$$

in which

$$K_L = \frac{c_L k_L}{(1 - t)^2} \; ; \quad K_R = \frac{c_R k_R}{(1 - t)^2} \qquad 52$$

Introducing (51) into (41) completes the formal proof of the existence of bound (2) for some c_R, c_L, k_R, $k_L \leqslant 1$ and some $t \leqslant ½$.

(4) In this section some simplified approximations for K_L and K_R are suggested for reliability practice. The value of $\varepsilon(t)$ gradually decreases as t decreases (see figure 9) and bound (3) becomes less conservative. Strong evidence suggests that (3) still holds true, when $t = 0$ in (52).

(i) When the variation of the load L is negligible compared to the strength variation ($\sigma_L = 0$, $L = m_L$), then, from (49)

$$P_f = P\left[R \le m_L\right] \le \frac{c_R k_R \sigma_R^2}{(m_R - m_L)^2} \qquad 53$$

(ii) When the strength is a deterministic quantity ($\sigma_R = 0$, $R = m_R$), then, from (51)

$$P_F = P\left[L \ge m_R\right] \le \frac{c_L k_L \sigma_L^2}{(m_R - m_L)^2} \qquad 54$$

Thus, when either σ_L or σ_R approaches zero, (3) is seen to hold for values of $K_L = c_L k_L$ and $K_R = c_R k_R$, i.e., $t = 0$.

(iii) When R and L are independent random variables (as is commonly assumed in safety analysis), then the variance of the safety margin $M = R - L$, equals the sum of the variances of R and L, $\sigma_M^2 = \sigma_R^2 + \sigma_L^2$. From (49), in which X is substituted by M and a by 0,

$$P_F = P\left[M \le 0\right] \le \frac{c_M k_M (\sigma_R^2 + \sigma_L^2)}{(m_R - m_L)^2} \qquad 55$$

Disregarding the effect of the factors c_R, c_L and c_M, it can be argued that, whenever $k_M \leqslant k_R = k_L$, t may again assume the value zero in (52). This follows from the fact that the values K_R and K_L may be chosen such that bound (55) equals bound (3). The central limit theorem indicates that, when L and R have unimodal density functions with skewness characteristics λ_R and λ_L, the density function of their difference tends to remain unimodal, but with a lower skewness characteristic λ_M. It is precisely these skewness characteristics which determine the factors k_R, k_L and k_M when the density functions are unimodal. In case nothing can be said about the distributations of R and L, then $k_R = k_M = 1$ again leads to a choice $t = 0$.

Thus, substituting c_L and c_R by unity, their upper bound, and putting $t = 0$ in (52), the values of K_L and K_R, to be used in (3) become $K_L = k_L$ and $K_R = k_R$. Note that, when the distribution of R is unimodal with mode $\mathcal{M}_R$

$$K_R = k_R = \frac{4}{9} \frac{(1 + \lambda_R^2)}{\left(1 - \frac{|\lambda_R| \sigma_R}{m_R - m_L}\right)^2} \simeq \frac{4}{9}(1 + \lambda_R^2) \qquad 56$$

because in practice

$$|\lambda_R| = \frac{|m_R - \mathcal{M}_R|}{\sigma_R} << \frac{m_R - m_L}{\sigma_R} \qquad 57$$

Similarly, when the distribution of L is unimodal with mode $\mathcal{M}_L$

$$K_L = k_L \simeq \frac{4}{9}(1 + \lambda_L^2) \qquad 58$$

because in practice

$$|\lambda_L| = \frac{|m_L - \mathcal{M}_L|}{\sigma_L} << \frac{m_R - m_L}{\sigma_L} \qquad 59$$

The Chebychev inequality guarantees that the values K_R and K_L do not exceed unity.

In summary it is suggested that Equation 3 be used with $K_R = 1$ if nothing is known about the distribution of R, and with $K_R = \frac{4}{9}(1 + \lambda_R^2)$ of R is known to have a unimodal distribution. Analogous conclusions hold for K_L.

APPENDIX III

The cross-sectional resistance R is a function of the random variables X_i

$$R = R(X_1, X_2, X_3, \ldots,)$$

A multidimensional Taylor expansion about the mean value m_{X_i} is suggested by observing the fact that the X_i are likely to lie close to m_{X_i} if their variability is not substantial.

$$R\ (X_1, X_2, \ \ldots\ ) = R(m_{X_1}, ,m_{X_2}, \ \ldots\ )$$

$$+ \sum_i (X_i - m_{X_i}) \frac{\partial R}{\partial X_i}\bigg|_{m_{X_i}} + \ldots \qquad 60$$

Taking the expectation of both sides

$$E(R) \cong R(m_{X_1}, m_{X_2}, \ \ldots)$$

since $E(X_i - m_{X_i}) = 0$. 61

Also, since $\mathrm{Var}\left[R(m_{X_1}, m_{X_2}, \ \ldots\ )\right] = 0$,

$$\mathrm{Var}\left[R\right] = \mathrm{Var}\left[\sum_i (X_i - m_{X_i}) \frac{\partial R}{\partial X_i}\bigg|_{m_{X_i}}\right]$$

$$= \sum_i \left(\frac{\partial R}{\partial X_i}\bigg|_{m_{X_i}}\right)^2 \mathrm{Var}\left[X_i\right] \qquad 62$$

$$+ \sum_i \sum_{j \neq i} \left(\frac{\partial R}{\partial X_j}\bigg|_{m_{X_j}}\right)\left(\frac{\partial R}{\partial X_j}\bigg|_{m_{X_j}}\right) \mathrm{cov}(X_i, X_j)$$

If the X_i are uncorrelated, then cov $(X_i, X_j) = 0$ $(i \neq j)$, and the approximation is simply

$$\mathrm{Var}\left[R\right] = \sum_i \left(\left. \frac{\partial R}{\partial X_i} \right|_{m_{X_i}} \right)^2 \mathrm{Var}\left[X_i\right] \qquad 63$$

This result merely states that the variability of the cross-sectional resistance is the net result of contributions of variability of all related parameters. The contribution of each component depends on its variability and on its relative importance in determining the overall resistance.