BEAMS ON ONE-WAY ELASTIC FOUNDATIONS

by

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Introduction

Mat foundations under certain structures, such as silos, water-storage tanks, coal-storage towers, and footing foundations supporting a group of columns, are frequently designed and constructed in the form of beams resting on soil. The theory of bending of beams on elastic foundations, developed by Winkler (1)[4], is based on the assumption that the intensity of the continuously distributed reaction of the foundation at every point is proportional to the deflection at that point. Its application to the design of foundations has received considerable attention. Other methods of analysis have been proposed by Hetenyi (2), Biot (3), Vesić (4), Levinton (5), Malter (6), Bowles (7), and Matlock (8). One common feature of these works is that the foundation can support a tensile stress. Recently, Tsai and Westmann (9) have indicated an approach based on a tensionless foundation assumption to account for the effects of beam uplift. A simplified procedure for the solution of the beam-foundation problem for the case of tensionless soil will be presented.

Problem Formulation

In the classical solution for beams on foundations, it is usual to assume that foundation properties are identical in tension and compression. Often the resulting analysis then indicates an alternating reaction, thus implying that the foundation can support a tensile stress. Usually this is not an acceptable result for real soil. Therefore, the Winkler model is modified to take into account the effect of beam uplift which then leads to a non-linear solution (9). As the beam is supported along its entire length by an elastic medium, which may or may not be continuous, the problem formulation and solution can be made by assuming that the beam rests on "one-way", equally spaced, elastic springs. The springs may be spaced in such a way and have the appropriate stiffnesses such that they adequately represent the soil medium. The subgrade tensile stress in the uplift portion of beam can be relaxed simply by setting the spring constants of those portions equal to zero.

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[4] Numerals in parentheses refer to corresponding items in Appendix I – References

Elastic solutions of beam-foundation problems are based on the assumption that the soil behaves as an elastic, homogeneous, infinite, and isotropic solid, defined by a modulus of deformation, E_s, and a Poisson's ratio, ν. It is further assumed that there are no shearing stresses at the contact interface between beam and soil. If the problem is so posed, Winkler's model can be replaced by a continuous beam resting on a set of springs each with stiffness constant K (10). This stiffness is defined by

$$K = K_s' \, a$$

where

$K_s' = K_s \; B$ = modulus of subgrade reaction x width of beam.

a = cell length (distance between springs equally spaced).

The solution of this problem then can be expressed by a matrix formulation as follows (11). Consider a beam supported by equally spaced springs, shown in Fig. 1, where γ is the uniform dead load, and Q is a concentrated load. In the following, each spring support point on the beam is considered to be a joint.

Load Matrix (P) and Displacement Matrix (X)

The joint load matrix [P] is defined as a column vector whose elements are the applied joint loads with the joints fixed. The displacement matrix [X] consists of the final displacements at the joints measured in the same directions as the loads. Referring to Fig. 2, the load matrix [P] is expressed by

$$[P] = \begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ P_{n_c+1} \\ \hline P_{n_c+2} \\ P_{n_c+3} \\ \cdot \\ \cdot \\ \cdot \\ P_{2n_c+2} \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\\1\end{matrix}}\right\} \text{moment terms} \\ \text{------------} \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\\1\end{matrix}}\right\} \text{force terms} \end{matrix} \qquad (1)$$

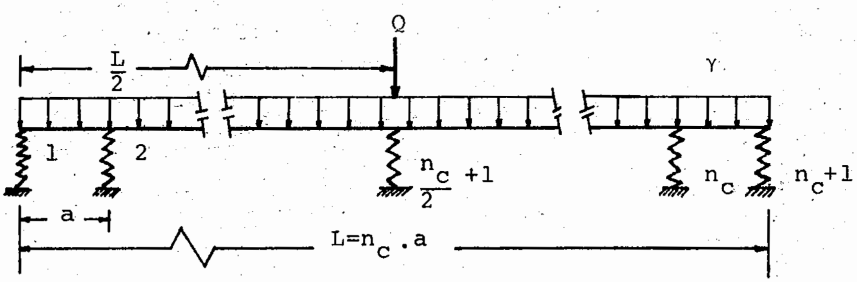

Fig. 1 Beam on Equally Spaced Spring Supports

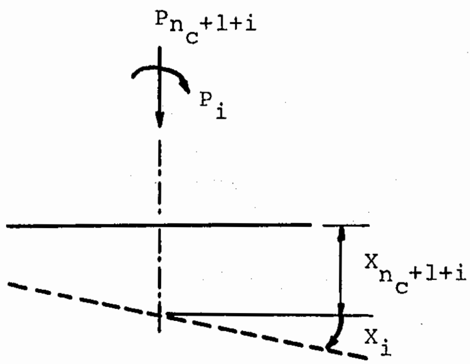

Fig. 2 Fixed Edge Forces and Final Deflections at ith Joint

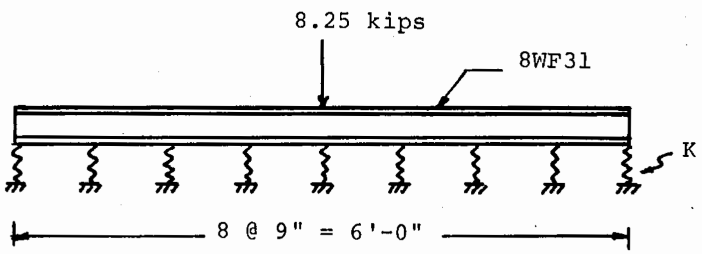

Fig. 3 Example 1. Beam tested by Vesić (4)

and the displacement matrix [X] is expressed by

$$[X] = \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_{n_c+1} \\ \hline X_{n_c+2} \\ X_{n_c+3} \\ \cdot \\ \cdot \\ \cdot \\ X_{2n_c+2} \end{bmatrix} \quad \begin{matrix} \left.\right\} \text{rotation terms} \\ \\ \left.\right\} \text{displacement terms} \end{matrix} \qquad (2)$$

These forces and displacements are related by (11):

$$[P]_{NP} = [ASA^T]_{NP \times NP}[X]_{NP} \qquad (3)$$

In the above, NP = $2n_c + 2$. (A) relates joint forces to member and spring forces and is (11)

$$[A] = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right], \ (2n_c+2) \times (3n_c+1)$$

and

$$[A_1] = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{bmatrix}, \ (n_c + 1) \times (2n_c)$$

$$[A_2] = [0], \ (n_c+1) \times (n_c+1)$$

$$[A_3] \begin{bmatrix} 1/a & 1/a & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ -1/a & -1/a & 1/a & 1/a & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/a & -1/a & 1/a & 1/a & . & . & . & 0 & 0 & 0 & 0 \\ . & & & & & & & & & & & & . \\ . & & & & & & & & & & & & . \\ . & & & & & & & & & & & & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1/a & -1/a & 1/a & 1/a \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & -1/a & -1/a \end{bmatrix}$$

$(n_c + 1)$ x $(2n_c)$

$$[A_4] = \begin{bmatrix} -1 & & & & & 0 \\ & -1 & & & & \\ & & -1 & & & \\ & & & . & & \\ 0 & & & & . & \\ & & & & & . \end{bmatrix} , \quad (n_c + 1) \text{ x } (n_c + 1)$$

$[S]$ is the structure stiffness matrix and is given by (11)

$$[S] = \left[\begin{array}{c|c} S_1 & S_2 \\ \hline S_3 & S_4 \end{array}\right] , \quad (3n_c + 1) \text{ x } (3n_c + 1)$$

$$\begin{bmatrix} \frac{4EI}{a} & \frac{2EI}{a} & 0 & 0 & . & . & . & 0 & 0 \\ \frac{2EI}{a} & \frac{4EI}{a} & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & \frac{4EI}{a} & \frac{2EI}{a} & . & . & . & 0 & 0 \\ 0 & 0 & \frac{2EI}{a} & \frac{4EI}{a} & . & . & . & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & \frac{4EI}{a} & \frac{2EI}{a} \\ 0 & 0 & 0 & 0 & . & . & . & \frac{2EI}{a} & \frac{4EI}{a} \end{bmatrix}$$

$(2n_c)$ x $(2n_c)$

$[S_2] = [S_3] = (n_c + 1) \times (n_c + 1)$

$$[S_4] = \begin{bmatrix} K & & & & \\ & K & & & \\ & & \cdot & & 0 \\ & & & \cdot & \\ 0 & & & & K \end{bmatrix} \qquad (n_c + 1) \times (n_c + 1)$$

In the above, EI is the flexural stiffness of the beam.

From (3),

$$[X] = [ASA^T]^{-1} [P] \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

After the spring deformations have been determined from equation (4), the spring forces may be obtained by

$$F_i = K X_{n_c + 1 + i} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

in which F_i is the force in the ith spring and X_{n_c+1+i} is the deflection in the ith spring.

In applying the above procedure to the problem of a beam on a tensionless foundation, an iterative technique is used. Deflections are first calculated as though the springs can take tension. For those points wherein the beam deflects upward the spring constants are set equal to zero in the $[S^4]$ matrix and new deflections are determined. The solution generally converges within two or three cycles. A computer program written in FORTRAN IV was used to perform the numerical calculations.

This program is a modification of one given by Wang (11) and is efficient and fairly compact. It may be readily adopted as a subroutine in a larger program. The program was executed on the IBM 360-50 computer at the KSU Computation Center but may be easily modified to run on smaller computers such as the IBM 1620 or 1130. The program – source deck, listing and sample problem – may be obtained from the authors.

Numerical Examples

Example 1.

A short beam (Fig. 3) which has its unit weight included in the analysis for the purpose of a comparison with results given from a soil test by Vesić (4).

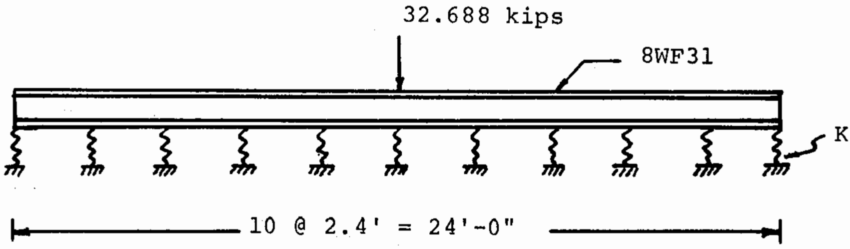

Fig. 4 Example 2A. Short Beam Comparison with Bowles Finite and Infinite Beam Analysis (7)

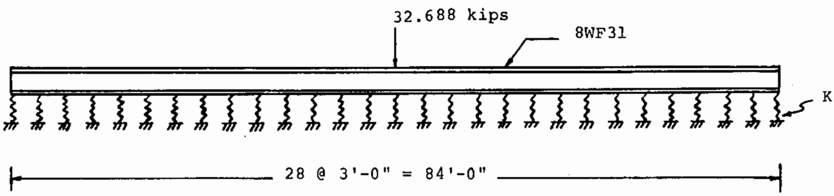

Fig. 5 Example 2B. Long Beam Comparison with Bowles Infinite Beam Analysis (7)

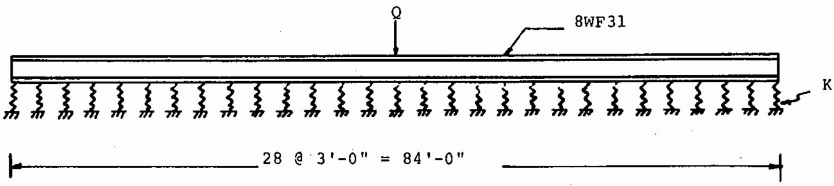

Fig. 6 Example 3. Long Beam Comparison with Infinite Beam Analysis of Tsai and Westmann (9) for Q = 8.6 kips, 12.9 kips, 17.2 kips, 34.4 kips

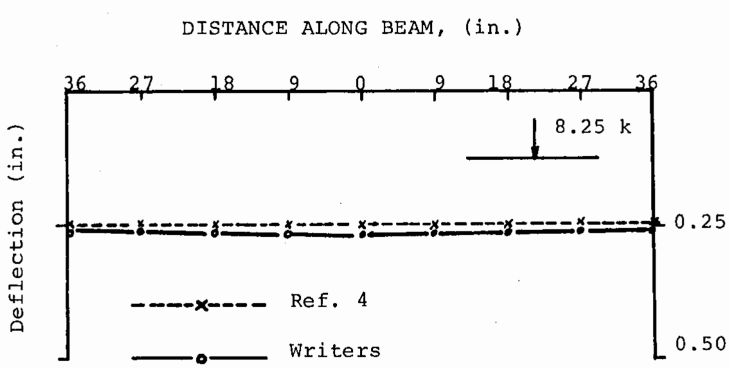

Fig. 7 Comparison with Soil Test Results (4) for Example 1

Beam length L = 72 in., center load Q = 8250 lb., unit weight γ = 31 lb./ft., spring constant K = 49 kip/ft. Cross section properties of the beam and subgrade are shown in Tables I and II. The results are shown in Fig. 7.

Table I – Data on Beam Section Used in Examples

Beam	Width B inches	Depth inches	Area Sq. in.	Moment Inertia I inch^4	Modulus of Elasticity E; psi
8WF31	8.0	8.0	9.12	109.7	30 x 10^5

Table II – Properties of Micaceous Silt Subgrade Used in Examples

Modulus of Elasticity of Soil Es, psi	Poisson's Ratio "‖"	Modulus of Subgrade Reaction Ks, psi
1192	0.25	454

Example 2.

Case A – A short beam with the same cross-section properties as in Example 1 but with unit weight not included in the analysis (Fig. 4). The results are compared with Bowles finite and infinite beam solutions (7) and plotted in Fig. 8. L = 2.4 ft., Q = 32.688 kips, K = 156.8 kip/ft.

Case B – The same beam as in Case A but longer and with the unit weight not included in the analysis (Fig. 5). The results are compared with Bowles infinite solution (7) and plotted in Fig. 9. L = 84 ft., Q = 32.688 kips, a = 3 ft., K = 196 kip/ft.

Example 3.

A long beam with the same cross-section as in Example 1 and with unit weight included in the analysis (Fig. 6). As the problem of the tensionless foundation is of prime interest, attention has been concentrated on solutions for the 8WF31 steel beam resting on a micaceous silt subgrade subjected to center loads of 8.6 kips, 12.9 kips, 17.2 kips, and 34.4 kips. This corresponds to the cases of n = 1.0, 1.5, 2.0, 4.0 discussed by Tsai (9). The results which are compared to his tensionless foundation solutions are shown in Figs. 10 to 13. Other parameters used in this example are L = 84 ft., a = 3 ft., γ = 31 lb./ft., K = 196 kip/ft.

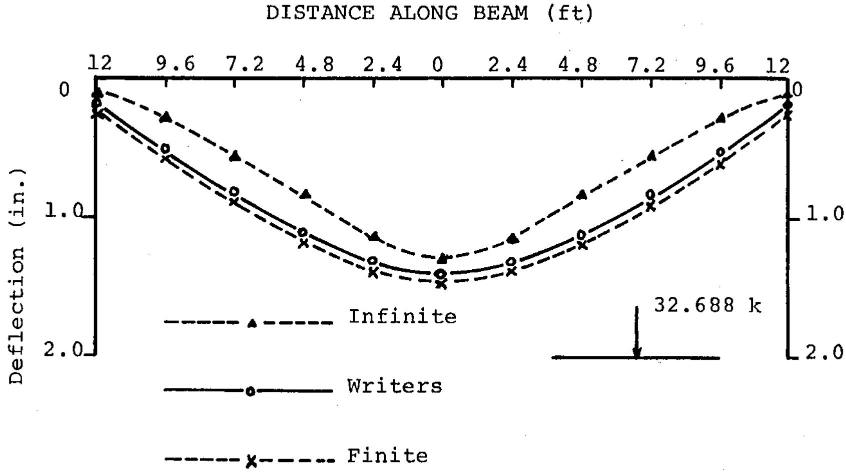

Fig. 8 Comparison of Deflections with Bowles (7) Finite and Infinite Beam Solution for Example 2A

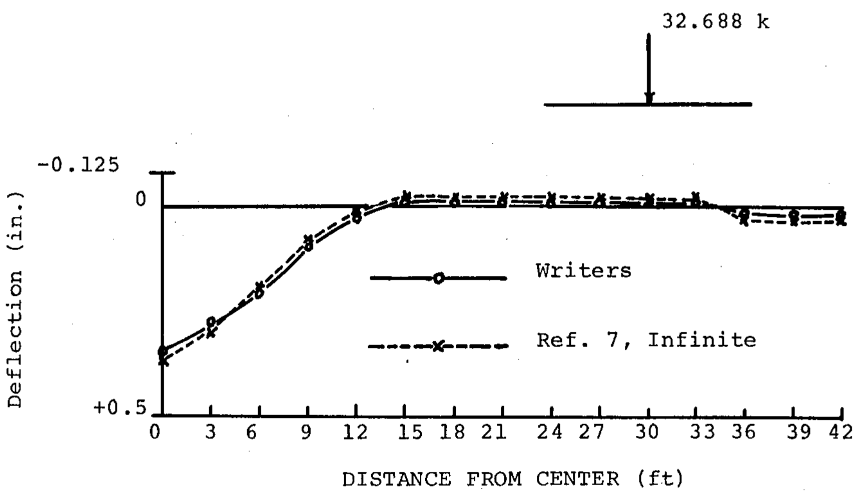

Fig. 9 Comparison of Deflections with Bowles (7) Infinite Beam Solution for Example 2B

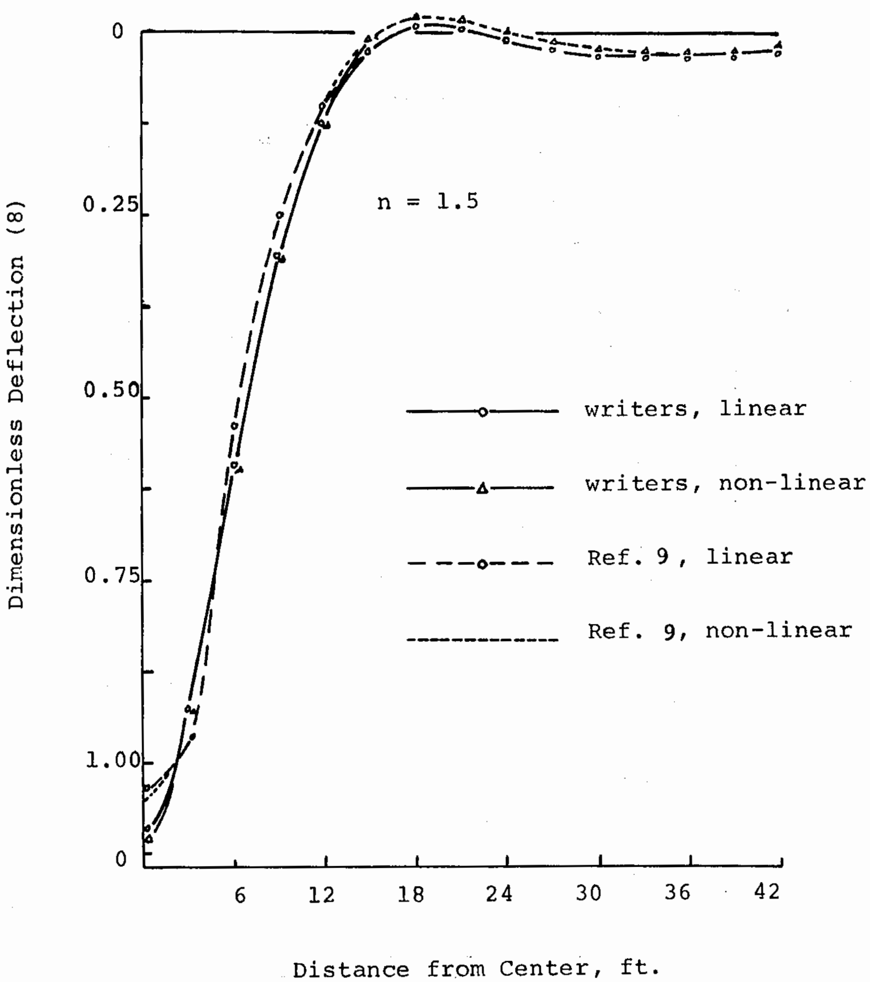

Fig. 10 Comparison of Linear (Tension Allowed) and Non-linear (Tensionless) Solutions for Example 3, $Q = 12.9^{k}$

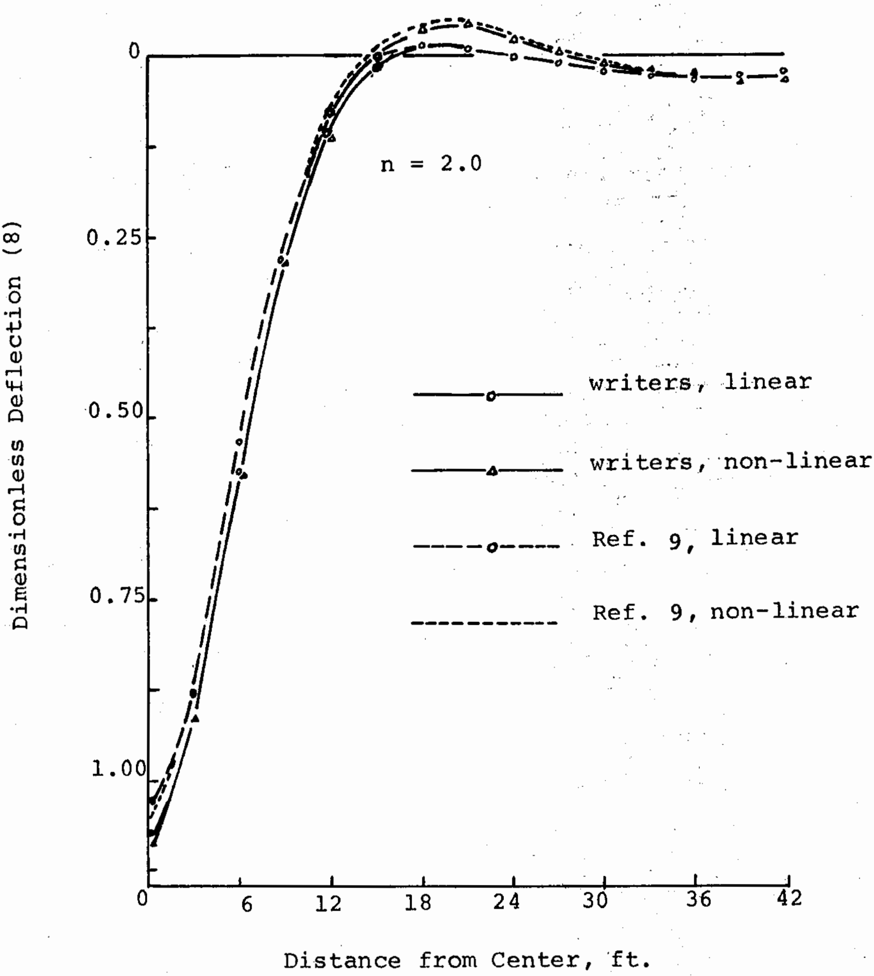

Fig. 11 Comparison of Linear (Tension Allowed) and Non-linear (Tensionless) Solutions for Example 3, $Q = 17.2^k$

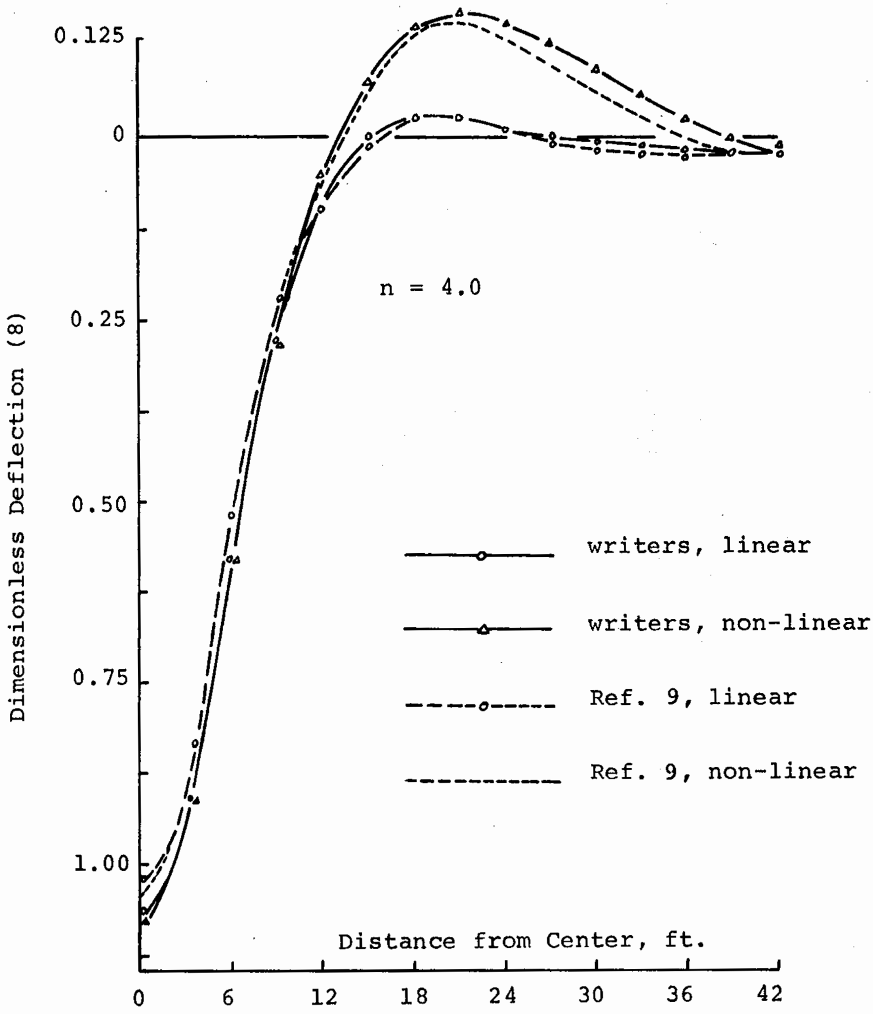

Fig. 12 Comparison of Linear (Tension Allowed) and Non-linear (Tensionless) Solutions for Example 3, $Q = 34.4^k$

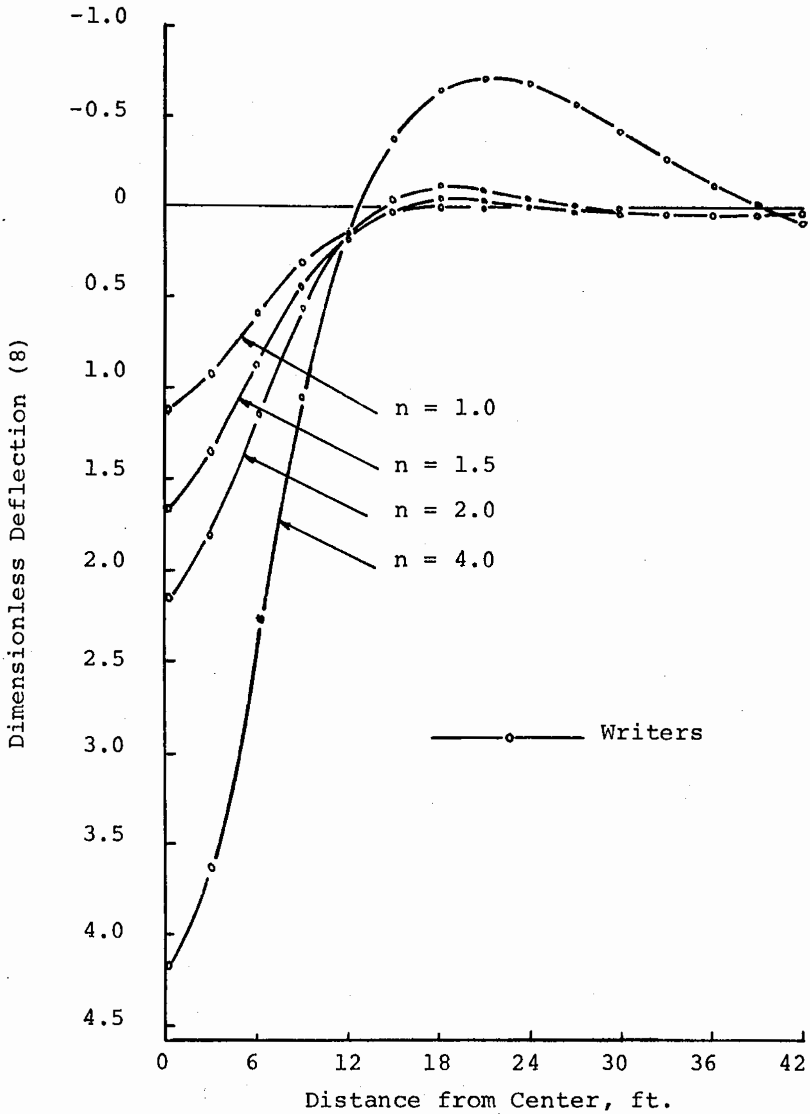

Fig. 13 Dimensionless Deflections, Summary of Load Cases for Example 3.

Summary and Conclusions

A method for determining the reaction pressures of beams subjected to dead weight and concentrated loads and supported on linearly elastic foundations which are unable to support tension has been presented. The supporting medium which is represented by a series of closely spaced springs may be continuous or discontinuous. The method may be readily modified to handle subgrades which exhibit non-linear force-deformation characteristics and/or non-homogeneous soil conditions. The method may also be extended quite easily to solve problems of mat or raft foundations subjected to numerous column loads and moments.

The following conclusions may be drawn:

1. The proposed method of analysis which is an application of a well-known technique in structural analysis is quite straight-forward and may be incorporated easily into a computer program.

2. The discrete solution presented is in good agreement with other solutions (4), (7), (9) for the cell lengths chosen in the examples. Better agreement may be expected if the cell lengths are decreased (more springs used) but the computer running time will increase.

3. Due to the non-linearity of the problem, superposition is not valid. This is illustrated in Example 3 (Fig. 13).

APPENDIX I. – REFERENCES

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APPENDIX II. – NOTATION

The following symbols are used in this paper:

A = transformation matrix
B = width of beam
E_s = modulus of elasticity of soil
EI = flexural rigidity of beam
K = spring constant
K_s = subgrade modulus
K_s' = K_sB subgrade modulus including the effect of the beam width
L = total length of beam
ν = Poisson's ratio
P = joint load matrix
Q = magnitude of concentrated center load
S = stiffness matrix
X = joint deformation matrix
a = cell length of beam-foundation
n = loading parameter
n_c = number of cells in the beam
γ = unit weight of beam