JOURNAL OF THE

BOSTON SOCIETY OF CIVIL ENGINEERS

Volume 52 | JANUARY, 1965 | Number 1

SOME FUNDAMENTAL CONCEPTS OF INCOMPRESSIBLE FLUID MECHANICS

PARTS I AND II

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(Presented as two of the John R. Freeman Lectures on Fundamental Hydraulic Processes in Water Resources Engineering, Boston Society of Civil Engineers, October 1 and 8, 1963.)

INTRODUCTION

The purpose of this lecture series as well as the accompanying notes is to present current thinking and recent research results in certain specialized fields of hydraulics. These particular paragraphs on fundamentals will therefore be restricted to what is hoped is a logical development of some of the background material to be drawn upon later. They make no attempt to avoid mathematics and assume a knowledge of basic hydraulics and the operations of partial differentiation.

I. FORMULATION OF BASIC EQUATIONS

Three basic laws which we bring to bear in the solution of problems of incompressible fluid mechanics are:

a. Conservation of mass.
b. Conservation of momentum.
c. Conservation of energy.

These laws can be formulated in two ways:

a. By focusing our attention on an imaginary closed surface in the fluid field (called a control surface) and considering the net

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flux (rate of flow) of mass momentum or energy across this surface. This will be called the control volume approach.

b. By identifying a particular mass particle of infinitesimal size and considering what happens to it in terms of time and space coordinates. This will be called the differential approach.

When the control volume is of differential size, the two methods become indistinguishable. The relative advantage of either approach must thus lie with the scale of the information desired. When changes in average flow properties between two locations are desired with no concern for the details of the flow in between, then the control volume approach is ideal. When the spatial distributions of flow properties are being sought, the differential approach is required.

Coordinate System

The coordinate system chosen for these developments is shown in Fig. 1.1. Unless specified otherwise, x is horizontal, z is vertical

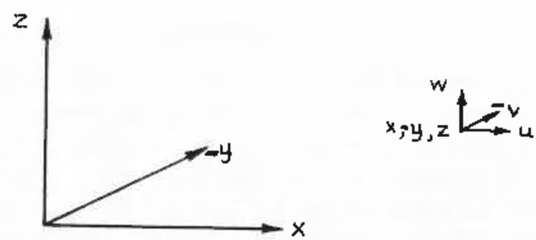

FIG. 1.1 COORDINATE SYSTEM

(positive upward) and y is perpendicular to the other two. At some instant of time, t, a fluid particle at x, y, z has the velocity components u, v, w, as is also shown in Fig. 1.1.

A. *Control Volume Approach*

Notation—

$\forall$ = control volume, ft^3
$d\forall$ = differential volume element, ft^3
A = control surface area, ft^2
dA = differential area element, ft^2

n = unit vector perpendicular to dA, positive outward
S = an extensive fluid property, i.e., some property possessed by each fluid element such as mass, linear momentum, energy, entropy, angular momentum, etc.
s = density of S or S per ft^3.

Let us consider a collection of particular fluid particles, those which completely fill the fixed boundaries of the control volume at time t_o. We will call this ensemble of particles the fluid system. If the fluid is flowing, the system and its cargo, S_s, will be convected across the control surface to some new position at later time, $t = t_o + \Delta t$. (Note: $S_{s_{t_o}} = S_{\overline{V}}$). The cargo of S carried by the system may change with time, first due to convection into a new location (a purely spatial change) and second due to some inherent unsteadiness in the flow (a purely local change).

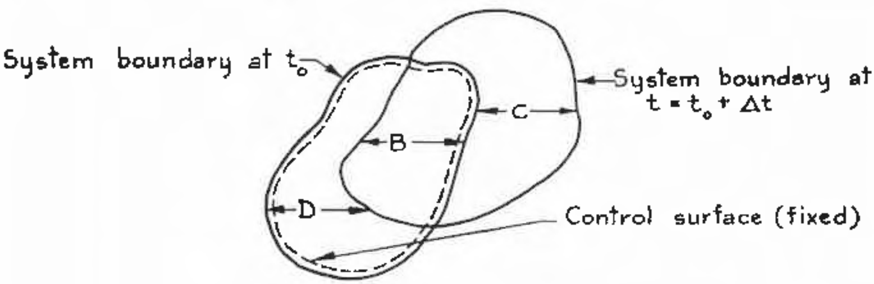

FIG. 1.2 CONTROL SURFACE AND SYSTEM BOUNDARIES

Referring to Fig. 1.2, the total time rate of change of system cargo may then be written:

$$\frac{dS_s}{dt} = \frac{S_{B_t} + S_{C_t} - S_{B_{t_o}} - S_{D_{t_o}}}{\Delta t}$$

or

$$\frac{dS_s}{dt} = \frac{S_{B_t} - S_{B_{t_o}}}{\Delta t} + \frac{S_{C_t} - S_{D_{t_o}}}{\Delta t}$$

Letting $\Delta t \rightarrow O$, $S_s \rightarrow S_{\overline{V}}$, and

$$\frac{dS_s}{dt} = \frac{dS_{\forall}}{dt} \equiv \frac{DS_{\forall}}{Dt} = \frac{\partial S_{\forall}}{\partial t}\bigg|_{t_0} + \text{net outflow rate of S} \tag{1.01}$$

The symbol D/Dt is often referred to as the total, substantial or material derivative and represents the total rate of change of some quantity experienced by the particular system of masses under consideration as the system moves through the control volume. The total amount of S within the control volume at any time is given by:

$$S_{\forall} = \int_{\forall} s \, d\forall$$

and the time rate of change of $S_{\forall}$ by

$$\frac{\partial S_{\forall}}{\partial t} = \frac{\partial}{\partial t} \int_{\forall} s \, d\forall \tag{1.02}$$

To evaluate the net rate of outflow of S from the control volume we may consider an element of surface area and the vector of the convecting velocity, q, at that point. Velocities directed outward from the control volume will be positive and inward will be negative.

Only the normal component, q_n, is effective in convecting S out through the control surface and thus the net rate of outflow of volume is

$$\int_A q_n \, dA$$

and the net rate of outflow of S is

$$\int_A s \, q_n \, dA \tag{1.03}$$

Thus in summary we can say:

Time rate of change of S for the system of masses =
Time rate of change of S within the control volume +
Net rate of outflow of S across the control surface.

Analytically

$$\frac{DS_s}{Dt} = \frac{\partial}{\partial t} \int_{\forall} s \, d\forall + \int_A s \, q_n \, dA \tag{1.04}$$

It should be noted that the quantity, s, may be a scalar such as energy or mass or a vector such as momentum as long as it depends upon the amount of fluid present.

Conservation of Mass

An example of the application of Eq. (1.04) is that in which the property, S, is mass. In this case $s = \rho$, the mass per unit volume.

Since system mass is to be conserved

$$\frac{DS_s}{Dt} = 0 = \frac{\partial}{\partial t}\int_{V} \rho \, dV + \int_{A} \rho \, q_n \, dA \tag{1.05}$$

Green's theorem tells us that

$$\int_{A} \rho \, q_n \, dA \equiv \int_{V} \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] dV$$

from which Eq. (1.05) can be written:

$$\int_{V} \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] dV = 0 \tag{1.06}$$

As a simple example of these ideas consider the one dimensional steady flow of Fig. 1.3 in which the control volume is chosen so that

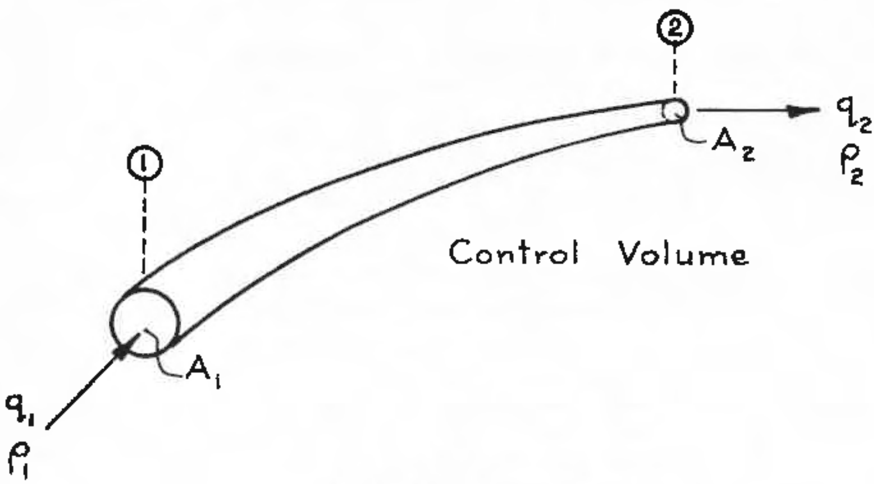

FIG. 1.3 STREAM TUBE

all elements of the lateral boundaries are streamlines for all time.

For this case Eq. (1.05) becomes:

$$\int_A \rho\, q_n\, dA = 0$$

or

$$\int_{A_1} \rho_1\, q_{1_n}\, dA + \int_{\text{side walls}} \rho\, q_n\, dA + \int_{A_2} \rho_2\, q_{2_n}\, dA = 0 \qquad (1.07)$$

Since the side walls are streamlines the middle integral of Eq. (1.07) is zero and we have

$$\int_{A_1} \rho_1\, q_1 \cos \alpha_1\, dA + \int_{A_2} \rho_2\, q_2 \cos \alpha_2\, dA = 0 \qquad (1.08)$$

Under the assumption of one-dimensional flow ρ, cos α and q are constant across each end section, thus

$$\rho_1\, V_1 \cos \alpha_1 \int_{A_1} dA + \rho_2\, V_2 \cos \alpha_2 \int_{A_2} dA = 0 \qquad (1.09)$$

If areas A_1 and A_2 are selected so as to be cross-sectional areas (i.e., normal to the mean velocities V_1 and V_2) we have

$$\cos \alpha_2 \int_{A_2} dA = A_2$$

and

$$\cos \alpha_1 \int_{A_1} dA = A_1$$

in which case we obtain the steady, one-dimensional equation of continuity applicable to either compressible or incompressible flows

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 \qquad (1.10)$$

The applicability of Eq. (1.10) is not restricted to truly one-dimensional flows, however, as long as ρ_1 and ρ_2 are constant across their respective sections and with the understanding that V_1 and V_2 are the components of the average velocity perpendicular to plane areas A_1 and A_2 respectively. For incompressible flows in which the lateral control surfaces are rigid boundaries Eq. (1.10) becomes

$$V_1 A_1 = V_2 A_2$$

which is applicable to steady or unsteady flow.

Conservation of Energy

Let us now consider the case in which S represents another scalar quantity, the fluid energy. Then

$$s = e\rho = \text{energy per unit volume}$$

in which e = energy per unit of mass. Eq. (1.04) then becomes

$$\frac{DE}{Dt} = \frac{\partial}{\partial t}\int_{\forall} e\rho \, d\forall + \int_{A} e\rho \, q_n \, dA \qquad (1.11)$$

We know that energy can enter or leave the system either in the form of heat or of work. If we let Q be the net heat added to the system and W be the net work done by the system on its surroundings, we have from the First Law of thermodynamics and Eq. (1.11) that:

$$\frac{DE}{Dt} = \frac{\delta Q}{dt} - \frac{\delta W}{dt} = \frac{\partial}{\partial t}\int_{\forall} e\rho \, d\forall + \int_{A} e\rho \, q_n \, dA \quad (1.12)$$

in which the symbol δ is used to indicate incremental amounts of items which are not system properties.

Looking first at the term for rate of doing net work on the surroundings, $\delta W/dt$:

Work can be done by the system on its surroundings only at the control surface:

a. where fluid contacts fluid, and

b. where fluid contacts solid.

Remember that the definition of work requires not merely that a force be present at the boundary, but also that this force move through a distance. In other words, the boundary must be in motion under the application of a force which can be resolved into components normal or tangential to the boundary surface.

The normal differential force on an element of the control surface is given in terms of the pressure intensity, p, by

$$dF_p = p \, dA \qquad (1.13)$$

and the rate at which work is done by these stresses is

$$\left.\frac{\delta W}{dt}\right|_p = \int_A p \, q_n \, dA = \int_A \rho\left(\frac{p}{\rho}\right) q_n \, dA \qquad (1.14)$$

The tangential differential force on an element of the control surface is given in terms of the shearing stress, τ, by

$$dF_{\tau} = \tau \, dA \tag{1.15}$$

and the rate at which work is done by these stresses is

$$\left.\frac{\delta W}{dt}\right|_{\tau} = \int_{A} \tau \, q_t \, dA \tag{1.16}$$

For the time being at least, due to the difficulty of evaluating this term, we will refer to it simply as the "shear power," P_{shear}.

If there are one or more rotating mechanical elements comprising a portion of the control surface (pumps, turbines, fans, compressors, etc.) it is convenient to exclude from Eqs. (1.14) and (1.16) those portions of the normal or tangential work which are performed on these elements (since they are excluded from the control volume they become part of the surroundings) and lump them separately in a term called "shaft power," P_{shaft}.

Thus:

$$\frac{\delta W}{dt} = P_{shaft} + P_{shear} + \int_{A} \rho \left(\frac{p}{\rho}\right) q_n \, dA \tag{1.17}$$

Looking next at the right hand side of Eq. (1.12) we must consider what forms of energy are to be included:

a. kinetic energy $= E_k = \dfrac{mq^2}{2}$

or $\quad e_k = \dfrac{q^2}{2}$ kinetic energy per unit mass,

b. gravitational potential energy $= E_g = mgz$

or $e_g = gz =$ gravitational potential energy per unit mass,

c. internal energy $= E_{U_*} = m\,u_*$

or $e_{U_*} = u_*$, internal energy per unit mass.

We will not consider here those energies due to electrical and magnetic fields, surface energies, etc. Thus

$$e = \frac{q^2}{2} + gz + u_* \tag{1.18}$$

and then

$$\frac{\delta Q}{dt} - P_{shaft} - P_{shear} = \frac{\partial}{\partial t}\int_{\forall} e\rho d\forall + \int_{A} \rho\left(\frac{q^2}{2} + gz + u_* + \frac{p}{\rho}\right) q_n\, dA \quad (1.19)$$

Steady, One-Dimensional Flow – ·

If we make the following simplifying assumptions:

a. Flow is steady, i.e., $\frac{\partial}{\partial t}\int_{\forall} e\rho d\forall = 0.$

b. Control surface coincides with solid, stationary boundaries or crosses the flow at right angles to the stream lines in zones of uniform flow where ρ and u_* are constant and where the pressure is "hydrostatically" distributed.

Then —

$$P_{shear} = 0$$

and

$$\int_{A} \rho\left(\frac{q^2}{2} + gz + u_* + \frac{p}{\rho}\right) q_n\, dA = \left[\left(k_e\frac{V^2}{2} + g\bar{z} + u_* + \frac{p}{\rho}\right)\rho VA'\right]_{\text{OUTFLOW}} - \left[\left(k_e\frac{V^2}{2} + g\bar{z} + u_* + \frac{p}{\rho}\right)\rho VA'\right]_{\text{INFLOW}}$$

in which

$$k_e = \frac{\int_{A'} \rho\, q^3\, dA'}{\rho\, V^3\, A'}.$$

Letting

$Q' =$ mass flow rate $= \rho VA'$,

$h =$ enthalpy per unit mass $= u_* + \frac{p}{\rho}$,

$q' = \frac{\delta Q}{dt}$,

$k_e = 1$,

and indicating the outflow and inflow sections by subscripts 2 and 1 respectively, Eq. (1.19) may be written:

$$q' = P_{shaft} + \left[h_2 - h_1 + \frac{V_2^2 - V_1^2}{2} + g(\bar{z}_2 - \bar{z}_1) \right] Q' \tag{1.20}$$

To better appreciate the significance of this simple form of the general energy equation let us look at some specific cases.

a. Suppose the shaft work is zero and we have a freely-falling jet of fluid. If the fluid is real we know that the mechanical energies as represented by $V^2/2$ and $g\bar{z}$ are not conserved. That is all of the loss of potential energy does not appear in increased kinetic energy. The missing energy has been degraded and will appear as a change in enthalpy, h and/or in heat transfer, q'.

b. Again for no shaft work if the fluid is real and incompressible and if the temperature is maintained a constant, the effects of fluid friction appear as a heat transfer, q'. This can be easily seen by considering flow in a horizontal conduit of uniform section. Since the internal energy u_* depends solely on temperature, Eq. (1.20) reduces to

$$q' = \left[\frac{p_2}{\rho} - \frac{p_1}{\rho} \right] Q'$$

or, more generally, for real, incompressible flows with no shaft work

$$\Delta_{1\,2} \left[\frac{p}{\rho} + \frac{V^2}{2} + g\bar{z} \right] Q' = \text{Rate of energy "loss"} = Q' \Delta u_* - q'$$

c. When the rate of energy "loss" is zero

$$q' = Q' \Delta u_*$$

and

$$\frac{p}{\rho} + g\bar{z} + \frac{V^2}{2} = \text{constant}$$

which is the familiar equation of Bernoulli.

Conservation of Momentum

We will now consider the case in which S represents a vector quantity, the linear fluid momentum, P. Then

$$\frac{s}{\rho} = \frac{P}{m} = q = \text{linear momentum per unit mass or velocity}$$

Equation (1.04) is then a vector equation:

$$\left[\frac{DP}{Dt}\right]_i = \frac{\partial}{\partial t}\int_{\forall} \rho\, q_i\, d\forall + \int_A \rho\, q_i\, q_n\, dA \qquad (1.21)$$

in which the subscript, i, refers to a particular coordinate direction.

We can expand the left hand side of Eq. (1.21) in terms of Newton's second law of motion which states, for a fixed (i.e. inertial) frame of reference:

$$F_i = \left(\frac{DP}{Dt}\right)_i \qquad (1.22)$$

or, in words — The resultant external force acting on a system of masses is equal to the total rate of change of linear momentum of the system.

Equation (1.21) then becomes

$$F_i \frac{\partial}{\partial t}\int_{\forall} \rho\, q_i\, d\forall + \int_A \rho\, q_i\, q_n\, dA \qquad (1.23)$$

Looking again at the stream tube of Fig. 1.3 we can write for one-dimensional, steady flow

$$F_i = \rho_1 V_1 V_{1_i} \cos \alpha_1 \int_{A_1} dA + \rho_2 V_2 V_{1_2} \cos \alpha_2 \int_{A_2} dA \qquad (1.24)$$

If A_1 and A_2 are selected normal to the mean velocities V_1 and V_2 and the density is uniform, Eq. (1.24) reduces to the familiar form

$$F_i = \rho V_2 A_2 V_{2_i} - \rho V_1 A_1 V_{1_i} = \rho VA \; (\Delta V_i) \qquad (1.25)$$

Another and independent vector equation similar to Eq. (1.21) can be written for conservation of angular momentum. For a discussion of this case see Shames (4).

B. *Differential Approach*

In order to study the details of the fluid motion at some point in space we will consider control volumes and mass elements of differential size.

Taylor's Series

If some fluid characteristic, f, such as velocity, pressure, temperature, density, etc., is known at point (x_o, y_o, z_o) and all spatial derivatives of f are known at this point, then the value of f at some new point can be determined. As an example, considering a variation, Δx, in the x coordinate direction, we can use Taylor's series to write

$$f(x_o + \Delta x, y_o, z_o) = f(x_o, y_o, z_o) + \left.\frac{\partial f(x, y, z)}{\partial x}\right|_{(x_0, y_0, z_0)} \frac{\Delta x}{1!} + \left.\frac{\partial^2 f(x, y, z)}{\partial x^2}\right|_{(x_0, y_0, z_0)} \frac{(\Delta x)^2}{2!} + \ldots \quad (1.26)$$

One important characteristic of this equation is that as we let Δx become very small, the quantities $(\Delta x)^2$, $(\Delta x)^3$, . . . become even smaller and Eq. (1.01) can be approximated by:

$$f(x_o + \Delta x, y_o, z_o) = f(x_o, y_o, z_o) + \left.\frac{\partial f(x, y, z)}{\partial x}\right|_{(x_0, y_0, z_0)} \Delta x \quad (1.27)$$

Conservation of Mass

Consider the small rectangular parallelepiped with sides of length Δx, Δy and Δz as shown in Fig. 1.4. Conservation of mass requires that the net mass of fluid flowing across the boundaries into the volume element in a certain time, Δt, be equal to the amount by which the mass of the element has increased in the same Δt.

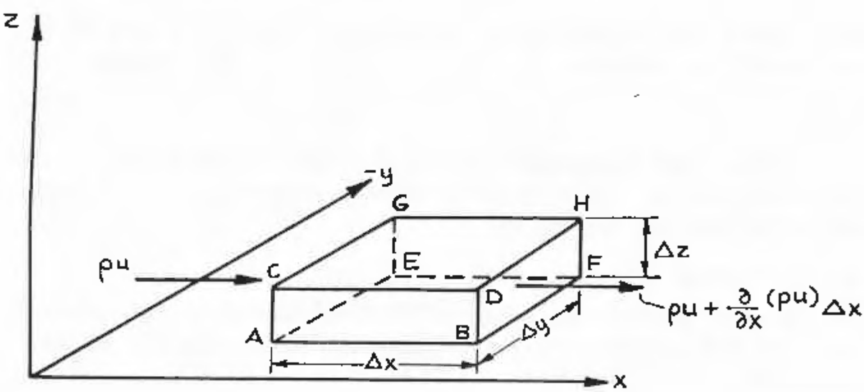

FIG. 1.4 FLOW INTO ELEMENT

The inflow of mass across face AECG in time Δt is

$$\rho u \Delta y \Delta z \Delta t$$

where ρ is the mass density of the fluid.

According to Taylor's series, the inflow of mass across the opposing face BFDH, in time Δt is

$$-\left[\rho u \, \Delta y \, \Delta z \, \Delta t + \frac{\partial(\rho u)}{\partial x} \Delta x \, \Delta y \, \Delta z \, \Delta t \right].$$

Adding the above two expressions yields the net inflow of mass in the x-direction during Δt:

$$-\frac{\partial(\rho u)}{\partial x} \Delta x \, \Delta y \, \Delta z \, \Delta t.$$

In a similar fashion it can be shown that the net inflow of mass during Δt in the y and z directions is respectively

$$-\frac{\partial(\rho v)}{\partial y} \Delta x \, \Delta y \, \Delta z \, \Delta t$$

$$-\frac{\partial(\rho w)}{\partial z} \Delta x \, \Delta y \, \Delta z \, \Delta t$$

The net inflow of mass into the volume element is the sum of the contributions of the three coordinate directions, i.e.:

Net inflow of mass across all faces in Δt

$$= \left[-\frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z} \right] \Delta x \, \Delta y \Delta z \, \Delta t \qquad (1.28)$$

If the mass present at time t is $\rho \Delta x \Delta y \Delta z$, then at time $t + \Delta t$, according to Taylor's formula, the mass present will be

$$\rho \Delta x \Delta y \Delta z + \frac{\partial}{\partial t} (\rho \Delta x \Delta y \Delta z) \, \Delta t.$$

The net increase is thus:

$$\frac{\partial \rho}{\partial t} \Delta x \Delta y \Delta z \Delta t \qquad (1.29)$$

In the absence of any creation or destruction of mass within the volume element, this must be equal to the inflow of mass across the boundaries, i.e., Eq. (1.28) = Eq. (1.29), or

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] \tag{1.30}$$

which is seen to be the differential form of Eq. (1.06).

The right hand side of this equation can be expanded to yield

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \tag{1.31}$$

Examining the first four terms of Eq. (1.31) we recognize the expanded form of the material derivative of the density, $D\rho/Dt$, in which

$$\frac{D}{Dt} \equiv \underbrace{\frac{\partial}{\partial t}}_{\text{local term}} + \underbrace{u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}}_{\text{convective terms}} \tag{1.32}$$

We can now see clearly that when focusing our attention on a particular particle we must consider the rate of increase of a certain quantity due to two effects:

(1) a "local" effect independent of the motion of the particle. This effect is the rate of change that a motionless particle would experience at a certain point.

(2) a "convective" effect which is the rate of change of the property due to the particle moving in a field where gradients of the property exist.

A simplified case which may clarify the physical significance of these effects is that of a person traveling across country by car. Consider that throughout the area there is a uniform increase in the mean daily temperature of 1 degree/day. This is the "local" effect and a person would experience this rate of temperature change even if he were not moving. In addition, suppose there is a mean daily temperature increase of 1 degree/1000 miles in the positive x direction. If the car is traveling in the x direction with velocity u then the *total* rate of increase experienced by the car would be

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = 1 + \frac{u}{1000}$$

where T is the mean daily temperature. Furthermore if the rate of travel, u, is 500 miles/day in the positive x direction, the total rate of increase experienced by the car is

$$\frac{DT}{Dt} = 1 + \frac{500}{1000} = 1.5 \text{ degrees/day}$$

Conversely, if one were traveling 1000 miles/day in the negative x direction, the total rate of temperature increase would be:

$$\frac{DT}{Dt} = 1 + (-1000)\frac{1}{1000} = 0$$

or the two effects would exactly compensate and one would experience no increase in the mean daily temperature.

Granted, now that D/Dt of a quantity does represent the total derivative or total rate of increase of that particular quantity experienced by a certain moving particle, let us return and express Eq. (1.04) in the form:

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \tag{1.33}$$

which is the continuity equation for a compressible or incompressible fluid in steady or unsteady motion. If we further idealize for the case of an incompressible fluid, the mass density of a particle must be constant, i.e., $D\rho/Dt = 0$ and hence Eq. (1.08) becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1.34}$$

Conservation of Momentum

When working in differential form, the momentum equations are very often called equations of motion. These equations will be derived for the case of a real (viscous) fluid of constant density by considering all of the forces which can act on the element of fluid shown in Fig. 1.5. Two types of forces must be considered, surface forces and body forces.

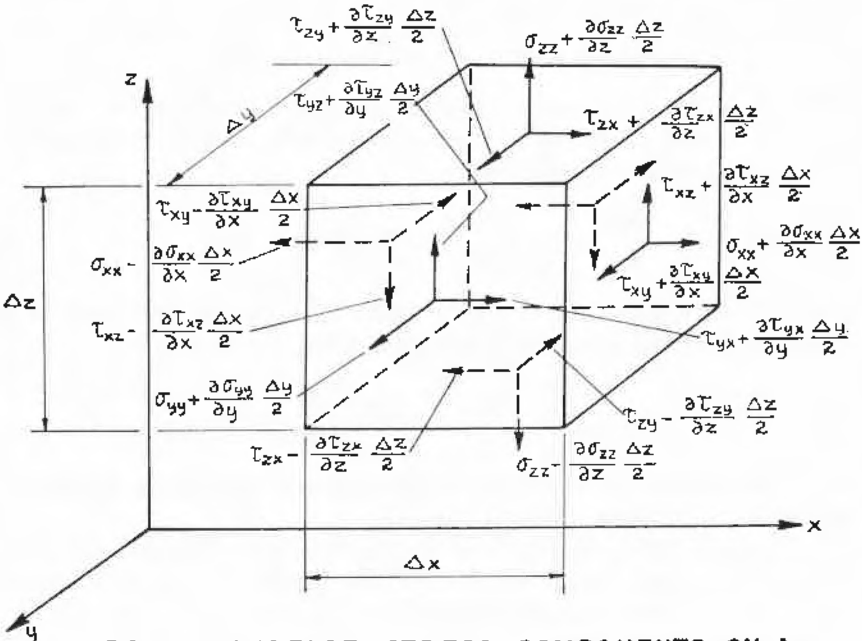

FIG. 1.5 SURFACE STRESS COMPONENTS ON A VOLUME ELEMENT

1. Surface forces—Any surface force may be resolved into components perpendicular and tangent to the surface. In terms of force per unit area, these become the direct stress, σ, and shear stress, τ, respectively.

In this development each stress component will bear two subscripts. The first defines the axis which is perpendicular to the element face upon which the stress acts, the second defines the coordinate direction in which the force produced by the stress acts. The sense of the stress is positive if both of these directions are positive or negative and negative if either one is negative. There are nine of these stress components.

2. Body forces—A body force depends only upon the bulk or mass of fluid included in the volume element. Gravity is the only body force to be considered in this study.

The nine surface stress components can be written as indicated in Fig. 1.5. The net normal force on the element in the positive x direction is thus

$$\left[\left(\sigma_{xx}+\frac{\partial\sigma_{xx}}{\partial x}\frac{\Delta x}{2}\right)-\left(\sigma_{xx}-\frac{\partial\sigma_{xx}}{\partial x}\frac{\Delta x}{2}\right)\right]\Delta y\Delta z=\frac{\partial\sigma_{xx}}{\partial x}\Delta x\Delta y\Delta z$$

and the net tangential force in the positive x direction is

$$\begin{aligned}\left[\left(\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}\frac{\Delta y}{2}\right)-\left(\tau_{yx}-\frac{\partial\tau_{yx}}{\partial y}\frac{\Delta y}{2}\right)\right]\Delta z\Delta x&\\ +\left[\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}\frac{\Delta z}{2}\right)-\left(\tau_{zx}-\frac{\partial\tau_{zx}}{\partial z}\frac{\Delta z}{2}\right)\right]\Delta x\Delta y&\\ =\left(\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)\Delta x\Delta y\Delta z&\end{aligned}$$

Designating X as the x component of the body force per unit of volume, the equation of motion in the x-direction can then be written:

$$d\,F_x=\left(\frac{DP}{Dt}\right)_x=\rho\frac{Du}{Dt}$$

surface forces + body forces = mass · acceleration

or

$$\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+\frac{\partial\sigma_{xx}}{\partial x}+X=\rho\frac{Du}{Dt}$$

and similarly for the other coordinate directions

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{zy}}{\partial z}+\frac{\partial\sigma_{yy}}{\partial y}+Y=\rho\frac{Dv}{Dt} \qquad (1.35)$$

and

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+Z=\rho\frac{Dw}{Dt}$$

Stresses and Strains

In a fashion similar to the above we may write equations of angular motion about coordinate axes through the mass center of the volume element. For example, adding moments about an axis parallel to the x axis and equating them to the moment of inertia times the angular acceleration, $\dot{\alpha}$, around this axis

$$\left[\left(\tau_{zy}+\frac{\partial\tau_{zy}}{\partial z}\frac{\Delta z}{2}\right)+\left(\tau_{zy}-\frac{\partial\tau_{zy}}{\partial z}\frac{\Delta z}{2}\right)\right]\Delta x\Delta y\frac{\Delta z}{2}$$
$$+\left[\left(\tau_{yz}+\frac{\partial\tau_{yz}}{\partial y}\frac{\Delta y}{2}\right)+\left(\tau_{yz}-\frac{\partial\tau_{yz}}{\partial y}\frac{\Delta y}{2}\right)\right]\Delta z\Delta x\frac{\Delta y}{2}$$
$$=\rho\,\Delta x\Delta y\Delta z\,\frac{(\Delta y^2+\Delta z^2)}{12}\,\dot{\alpha} \quad (1.36)$$

In the limit, as Δx, Δy and Δz are shrunk to zero, the right hand side of Eq. (1.36) must vanish and we have

$$\tau_{yz}=\tau_{zy}$$

Repeating this about the other centroidal axes yields

$$\tau_{yx}=\tau_{xy} \quad (1.37)$$

and

$$\tau_{xz}=\tau_{zx}$$

which reduces the number of scalar surface stress components to six

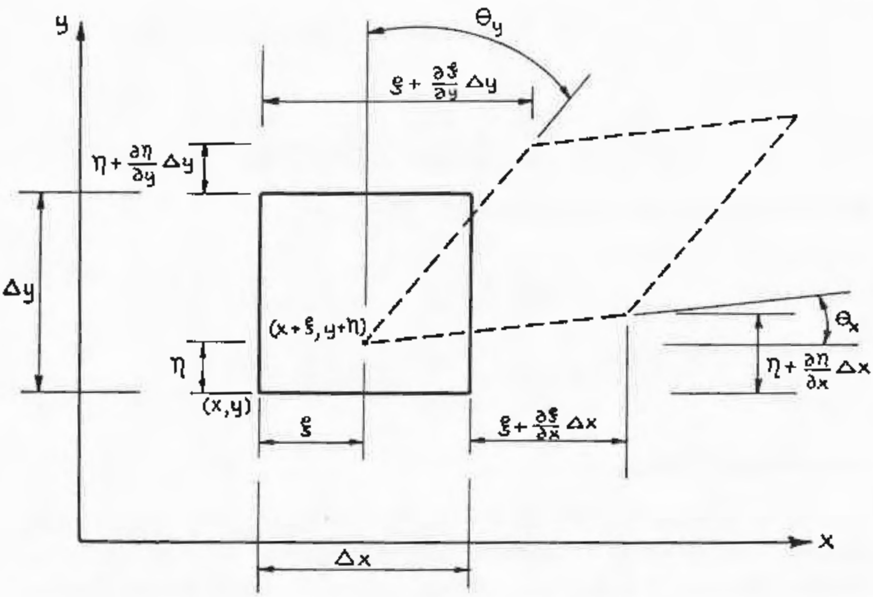

FIG. 1.6 PLANE STRAIN AND DEFORMATION

The negative average of the three direct stresses is that quantity which we know as fluid pressure, p. Thus

$$p = -\bar{\sigma} = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \tag{1.38}$$

Stress-Strain Relations

When stresses are applied to a volume element of any continuous medium, whether it be solid, liquid or gas, the element will deform. Let the coordinates of a point before deformation be given by x, y, z and after deformation by $x + \xi$, $y + \eta$, $z + \zeta$. This is illustrated for two dimensions in Fig. 1.6.

Defining normal strain as the change in length of a side of the element divided by its original length, the normal strain component, ε_x, can be written (see Fig. 1.6):

$$\varepsilon_x = \frac{\xi + \frac{\partial \xi}{\partial x}\Delta x - \xi}{\Delta x} = \frac{\partial \xi}{\partial x}.$$

Similarly

$$\varepsilon_y = \frac{\partial \eta}{\partial y} \tag{1.39}$$

and

$$\varepsilon_z = \frac{\partial \zeta}{\partial z}$$

The volumetric strain (dilatation), e, is then given by:

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z \tag{1.40}$$

Defining shear strain as the change in angle between pairs of axes perpendicular to surfaces of the undeformed element, the shear strain component, γ_{xy}, can be written (see Fig. 1.6):

$$\gamma_{xy} = \theta_x + \theta_y = \frac{\frac{\partial \eta}{\partial x}\Delta x}{\Delta x} + \frac{\frac{\partial \xi}{\partial y}\Delta y}{\Delta y} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}$$

Similarly

$$\gamma_{yz} = \frac{\partial \zeta}{\partial y} + \frac{\partial \eta}{\partial z} \tag{1.41}$$

$$\gamma_{zx} = \frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x}$$

Elastic Solid

When the medium is an elastic solid, we know from experiment that Hooke's law provides a linear proportionality between stress and the magnitude of the strain. This proportionality is defined by the shear modulus of elasticity, G, the bulk modulus of elasticity, E, and their interrelationship through the Poisson ratio, n, where

$$n = \frac{\text{unit strain in direction normal to applied force}}{\text{unit strain in direction of applied force}} = \frac{E}{2G} - 1 \tag{1.42}$$

Using these relations we can write, for an isotropic elastic solid:

$$\left.\begin{aligned} \varepsilon_x &= \frac{\sigma_{xx}}{E} - n\frac{\sigma_{yy}}{E} - n\frac{\sigma_{zz}}{E} = \frac{1}{E}\left[\sigma_{xx} - n(\sigma_{yy} + \sigma_{zz})\right] \\ \varepsilon_y &= \frac{1}{E}\left[\sigma_{yy} - n(\sigma_{zz} + \sigma_{xx})\right] \\ \varepsilon_z &= \frac{1}{E}\left[\sigma_{zz} - n(\sigma_{xx} + \sigma_{yy})\right] \end{aligned}\right\} \tag{1.43}$$

and

$$\left.\begin{aligned} \gamma_{xy} &= \frac{\tau_{xy}}{G} \\ \gamma_{yz} &= \frac{\tau_{yz}}{G} \\ \gamma_{zx} &= \frac{\tau_{zx}}{G} \end{aligned}\right\} \tag{1.44}$$

From Eqs. (1.40) and (1.43) the volumetric strain becomes

$$e = \frac{1-2n}{E}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \tag{1.45}$$

Equations (1.38), (1.42), (1.43), (1.44) and (1.45) can now be combined to give

$$\left.\begin{aligned} \sigma_{xx} - \bar{\sigma} &= 2G\left[\frac{\partial \xi}{\partial x} - \frac{1}{3}\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}\right)\right] \\ \sigma_{yy} - \bar{\sigma} &= 2G\left[\frac{\partial \eta}{\partial y} - \frac{1}{3}\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}\right)\right] \\ \sigma_{zz} - \bar{\sigma} &= 2G\left[\frac{\partial \zeta}{\partial z} - \frac{1}{3}\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}\right)\right] \end{aligned}\right\} \tag{1.46}$$

$$\left.\begin{aligned}\tau_{yx} = \tau_{xy} &= G\left(\frac{\partial\eta}{\partial x} + \frac{\partial\xi}{\partial y}\right)\\ \tau_{zy} = \tau_{yz} &= G\left(\frac{\partial\zeta}{\partial y} + \frac{\partial\eta}{\partial z}\right)\\ \tau_{xz} = \tau_{zx} &= G\left(\frac{\partial\xi}{\partial z} + \frac{\partial\zeta}{\partial x}\right)\end{aligned}\right\} \tag{1.47}$$

The units of the shear modulus of elasticity G are seen to be those of stress per unit of strain which is equivalent to stress since the strain is dimensionless.

Newtonian Fluids--

Experiments have shown that for fluids, stresses are related to the time rate of strain rather than to the strain magnitude as in the case of the solid. In analogy with Hooke's law, the most simple form of this relationship is the simple linear proportion

$$\frac{\text{stress}}{\text{time rate of strain}} = \mu \tag{1.48}$$

in which μ is a fluid property called the dynamic viscosity and has the units of stress · time. Fluids which obey this simple proportion are called Newtonian fluids.

Eqs. (1.46) and (1.47) can now be rewritten for Newtonian fluids by replacing G by μ and by taking the time derivative of all the strains. For example:

$$\sigma_{xx} - \bar{\sigma} = \sigma_{xx} + p = 2\mu\frac{\partial}{\partial t}\left[\frac{\partial\xi}{\partial x} - \frac{e}{3}\right]$$

Interchanging the order of differentiation and recognizing that

$$u = \frac{\partial\xi}{\partial t},\ v = \frac{\partial\eta}{\partial t},\ w = \frac{\partial\zeta}{\partial t}$$

we can write

$$\left.\begin{aligned}\sigma_{xx} + p &= 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\frac{\partial e}{\partial t}\\ \sigma_{yy} + p &= 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\frac{\partial e}{\partial t}\\ \sigma_{zz} + p &= 2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\frac{\partial e}{\partial t}\end{aligned}\right\} \tag{1.49}$$

and

$$\left.\begin{aligned}
\tau_{xy} &= \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\
\tau_{zy} &= \tau_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\
\tau_{xz} &= \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)
\end{aligned}\right\} \tag{1.50}$$

Equation of Motion for Newtonian Fluid

Substituting Eqs. (1.48) and (1.49) into Eqs. (1.35) we obtain

$$\left.\begin{aligned}
\rho\frac{Du}{Dt} &= X - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial e}{\partial t}\right)\right] \\
&+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] \\
\rho\frac{Dv}{Dt} &= Y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial e}{\partial t}\right)\right] \\
&+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \\
\rho\frac{Dw}{Dt} &= Z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\frac{\partial e}{\partial t}\right)\right] \\
&+ \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]
\end{aligned}\right\} \tag{1.51}$$

which are the famous Navier-Stokes equations which govern the dynamic behavior of Newtonian fluids. Along with these three equations we also have the equation of conservation of mass already derived:

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{1.30}$$

In the case of compressible fluids it is necessary to add an equation of state (relating pressure, density and temperature, T) an equation of energy (relating mechanical work and temperature distribution) and an empirical relation between viscosity and temperature. This gives seven equations in the variables u, v, w, p, ρ, T, μ.

For the incompressible case of primary interest here, the density of a given fluid particle is a function of neither time nor spatial position, thus:

1. The conservation of mass relation becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\equiv\frac{\partial e}{\partial t}=0 \tag{1.34}$$

2. Pressure variations will not cause large temperature changes through compression, thus the isothermal conditions necessary for the viscosity to be spatially invariant are more commonly approached.

Under these conditions the equations of motion for an isothermal incompressible Newtonian fluid can be written

$$\left.\begin{aligned}\rho\frac{Du}{Dt}&=X-\frac{\partial p}{\partial x}+\mu\nabla^2 u\\ \rho\frac{Dv}{Dt}&=Y-\frac{\partial p}{\partial y}+\mu\nabla^2 v\\ \rho\frac{Dw}{Dt}&=Z-\frac{\partial p}{\partial z}+\mu\nabla^2 w\end{aligned}\right\} \tag{1.52}$$

in which

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \tag{1.53}$$

Equations (1.34) and (1.52) provide 4 relations in the 4 variables u, v, w, p and constitute the theoretical foundation for incompressible, isothermal fluid mechanics. Unfortunately these equations are so complex that no general solution of them has been found. Particular solutions have been found, however, for certain special cases but before looking at these it is instructive to examine the dimensionless form of Eqs. (1.52).

Dimensionless Form of Equations of Motion

We will restrict ourselves to the case in which the only body force is that of gravity. We know that the gravitational force field is a potential field, that is, there exists a scalar function of space, Ω, such that with the axis z oriented vertically upward:

$$\left.\begin{aligned}\frac{\partial\Omega}{\partial x} &= -X = 0\\ \frac{\partial\Omega}{\partial y} &= -Y = 0\\ \frac{\partial\Omega}{\partial z} &= -Z = -\rho g\end{aligned}\right\} \quad (1.54)$$

in which g is the local gravitational constant.

Let us define a reference value for each basic variable:

$$l_o = \text{reference length}$$
$$V_o = \text{reference velocity}$$
$$l_o/V_o = \text{reference time.}$$

It is convenient to decompose the pressure intensity, p, into a static and a dynamic component, the static component, p_s, being that which would be present if the fluid was suddenly frozen and the dynamic pressure, p_d, that deviation from the static due to fluid acceleration, i.e.

$$p = p_s + p_d \quad (1.55)$$

in which p_s is written in terms of the "static" piezometric head, h_s, as

$$p_s = \gamma(h_s - z). \quad (1.56)$$

There will be two reference pressures:

$$\rho V_o^2 = \text{reference dynamic pressure}$$

and

$$\gamma l_o = \text{reference static pressure.}$$

To be selected as a reference value, the quantity should be important in the flow problem. For example, l_o might be the thickness of a bridge pier or the wave height in deep water. Using the reference values we can define the dimensionless variables:

$$\left.\begin{aligned} U &= u/V_o & l_x &= x/l_o & H_s &= h_s/l_o\\ V &= v/V_o & l_y &= y/l_o & P_d &= p_d/\rho V_o^2\\ W &= w/V_o & l_z &= z/l_o & T &= tV_o/l_o \end{aligned}\right\} \quad (1.57)$$

Substituting Eqs. (1.54) through (1.57) into Eqs. (1.34) and (1.52) gives:

$$\left.\begin{aligned}\frac{DU}{DT} &= -\frac{\partial P_d}{\partial l_x} - \mathbb{F}^{-2}\frac{\partial H_s}{\partial l_x} + \mathbb{R}^{-1}\nabla^2 U\\ \frac{DV}{DT} &= -\frac{\partial P_d}{\partial l_y} - \mathbb{F}^{-2}\frac{\partial H_s}{\partial l_y} + \mathbb{R}^{-1}\nabla^2 V\\ \frac{DW}{DT} &= -\frac{\partial P_d}{\partial l_z} + \mathbb{R}^{-1}\nabla^2 W\end{aligned}\right\} \tag{1.58}$$

$$\frac{\partial U}{\partial l_x} + \frac{\partial V}{\partial l_y} + \frac{\partial W}{\partial l_z} = 0 \tag{1.59}$$

in which

$$\left.\begin{aligned}\mathbb{F} &= \text{Froude number} = \frac{V_o}{\sqrt{gl_o}} = \frac{\text{inertia forces}}{\text{gravity forces}}\\ \mathbb{R} &= \text{Reynolds number} = \frac{V_o l_o \rho}{\mu} = \frac{\text{inertia forces}}{\text{viscous forces}}\end{aligned}\right\} \tag{1.60}$$

We have introduced a new variable, H_s, and thus need a new equation in order to have enough information available for problem solution. This equation comes from the statement that a fluid particle lying in the static piezometric surface, h_s, must always remain there. If the equation of this surface is

$$F_{h_s}(x, y, z, t) = 0 \tag{1.61}$$

this condition is written

$$\frac{DF_{h_s}}{Dt} = 0$$

Dynamic Similarity

We are now in a position to answer the very important question—What are the necessary and sufficient conditions for dynamic similarity?

By dynamic similarity we mean that condition under which the vector polygons of both force and velocity (the latter of these implies kinematic similarity) are geometrically similar at corresponding points in two geometrically similar systems.

In terms of Eqs. (1.58), (1.59) and (1.62) this means

(1) for the same values of T, l_x, l_y, l_z the two systems will have identical U, V, W, P_d and H_s.

(2) Corresponding terms in the dynamical equations (Eqs. 1.58) must be equal in the two systems.

These conditions are both met:

(1) By assuring geometric similarity of the two systems.

(2) By observing the time scale specified by equal T's in the two systems.

(3) By making the dynamic coefficients, $\mathbb{F}$ and $\mathbb{R}$, of Eqs. (1.58) equal in the two systems. (Note: If the original formulation of the Navier-Stokes equations had included other forces such as surface tension etc., other dynamic coefficients would be present.)

For closed systems the "static" piezometric head is everywhere constant and the problem solution becomes independent of H_s and $\mathbb{F}$.

For open systems (containing a free surface or other interface), H_s may vary and thus $\mathbb{F}$ is important as well as $\mathbb{R}$.

Problems in fluid mechanics are now reduced to determining, by theory, experiment or both, one or more of the dependent variables, U, V, W, P_d, H_s as functions of the independent parameters l_x, l_y, l_z, T and of the dynamic coefficients $\mathbb{F}$ and $\mathbb{R}$. Familiar examples are the many drag coefficient plots which present:

$$P_d = \frac{p_d}{\rho V_o^2} = \frac{F_D/A}{\rho V_o^2} = \frac{C_D}{2} = \phi(\mathbb{R}, \text{geometry})$$

or velocity distribution plots which usually show

$$U = \frac{u}{U_{max}} = \phi\left(\frac{z}{l_o}, \mathbb{R}\right)$$

etc.

Conservation of Energy

The differential form of the energy equation will not be developed here. The interested reader is referred to Rouse (3) for this material.

II. Reduction of Basic Equations in Special Cases

Rotationality vs. Irrotationality

Returning to the two-dimensional representation of Fig. 1.6 (p. 18) we note that one of the possible responses of a fluid particle to stress is a rotation of the particle about its mass center. This rotation is defined as the average angular velocity of two infinitesimal orthogonal line elements which lie in a plane perpendicular to the axis about which the rotation is desired. Analytically, from Fig. 2.1 the mean rate of change of angles α and β can be written:

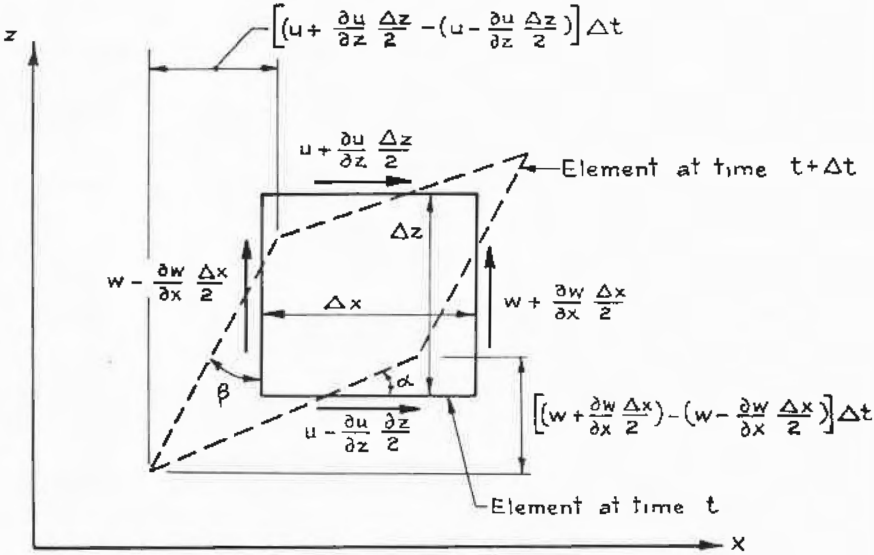

FIG 2.1 ROTATION OF A FLUID PARTICLE

$$\frac{\Delta\alpha}{\Delta t}=\frac{\left[\left(w+\frac{\partial w}{\partial x}\frac{\Delta x}{2}\right)-\left(w-\frac{\partial w}{\partial x}\frac{\Delta x}{2}\right)\right]\Delta t}{\Delta x\,\Delta t}=\frac{\partial w}{\partial x}$$

$$\frac{\Delta\beta}{\Delta t}=\frac{-\left[\left(u+\frac{\partial u}{\partial z}\frac{\Delta z}{2}\right)-\left(u-\frac{\partial u}{\partial z}\frac{\Delta z}{2}\right)\right]\Delta t}{\Delta z\,\Delta t}=-\frac{\partial u}{\partial z}$$

in which counter-clockwise rotations are taken to be positive. The mean angular velocity about the y axis is thus

$$\omega_y=\frac{1}{2}\left[\frac{\Delta\alpha}{\Delta t}+\frac{\Delta\beta}{\Delta t}\right]=-\frac{1}{2}\left[\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right] \quad (2.01)$$

in which the bracketed quantity is termed the vorticity. In order for a fluid to possess vorticity there must be a rotation of the particle. For this reason, flows in which the vorticity is zero or negligible are called irrotational flows.

Euler Equations

Making use of the two-dimensional equation of continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{2.02}$$

we can show that

$$\nabla^2 u = \frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]$$

and

$$\nabla^2 w = \frac{-\partial}{\partial x}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right] \tag{2.03}$$

thus, if we specify a flow to have constant vorticity, $\nabla^2 u = \nabla^2 w = 0$ and, regardless of the fluid viscosity, the isothermal, incompressible equations of motion for a Newtonian fluid (Eqs. 1.52) reduce to (for two dimensions):

$$\rho \frac{Du}{Dt} = X - \frac{\partial p}{\partial x}$$

$$\rho \frac{Dw}{Dt} = Z - \frac{\partial p}{\partial z} \tag{2.04}$$

These equations are called the Euler equations. It is seen from Eqs. (1.52) that these same equations apply exactly to the hypothetical case of a zero viscosity fluid regardless of the intensity or distribution of vorticity.

We must ask ourselves whether Eqs. (2.04) have any practical use. Is the vorticity ever constant or nearly so? If so, where and under what circumstances?

Origin and Distribution of Vorticity

It is obvious that a fluid at rest has zero vorticity everywhere. The fluid in all systems was once at rest thus if a fluid has vorticity now where did it come from? Since body forces act through the mass center of a fluid particle they can produce no rotation. Vorticity must thus result from the application of surface forces. Such forces are by definition applied only at the boundaries of the fluid system. Furthermore, when we compare Eqs. (1.50) and (2.03) we find that in certain cases zero shear stress in a real fluid means zero vorticity. In these cases only those boundaries at which a shear stress exists can generate vorticity. We recognize the solid boundary as just such a vorticity generator due to the "no slip" condition which a real fluid satisfies at these surfaces. Let's look at the two-dimensional distribution of vorticity near a surface of this kind in a flow having an ambient velocity, U_o.

Eliminating pressure between the first and last of Eqs. (1.52) and using Eqs. (2.01) and (2.02) we obtain the equation of vorticity transport

$$\frac{D\omega_y}{Dt} = \nu \nabla^2 \omega_y \tag{2.05}$$

in which $\nu = \frac{\mu}{\rho}$ = kinematic fluid viscosity. Eq. (2.05) is the differential form of a conservation of angular momentum relation and tells us that the total rate of change of vorticity of a fluid particle equals the rate at which vorticity is dissipated by friction. The left hand side of this equation represents convection of vorticity while the right hand side represents conduction at a rate governed by the molecular momentum diffusivity, ν, of the fluid. Since at this time we are only interested in an approximate solution to Eq. (2.05) we will make some very crude assumptions which are totally unacceptable in a quantitative sense but which will not distort the qualitative nature of the phenomenon. These are:

(1) The vertical velocity, w, is zero everywhere.

(2) Vorticity gradients in the z direction are much larger than in the x direction.

With these approximations Eq. (2.05) becomes, for steady flow

$$\nu \frac{\partial^2\omega_y}{\partial z^2} = u \frac{\partial\omega_y}{\partial x} \tag{2.06}$$

We will now assume that $\partial\omega_y/\partial x = -k_1 \frac{\partial\omega_y}{\partial z}$. The minus sign is crucial but logical. In the absence of w the z transport of vorticity is solely by diffusion and it is the fundamental nature of diffusive processes to have first and second derivatives of opposite sign. With this assumption Eq. (2.06) may be integrated once to obtain

$$\ln\left(\frac{\partial \omega_y}{\partial z}\right) = -\frac{k_1}{\nu}\int u\,dz + f_1(x)$$

which may be approximated

$$\frac{\partial \omega_y}{\partial z} = f_1(x)\,e^{-k_2 U_o z/\nu}$$

Integrating again

$$\omega_y \cong -\frac{1}{2}\frac{\partial u}{\partial z} = f_2(x) - f_3(x)\,e^{-k_2 U_o z/\nu} \qquad (2.07)$$

and again

$$u = \frac{-2\nu}{k_2 U_o} f_3(x)\,e^{-k_2 U_o z/\nu} - 2f_2(x)\,z + f_4(x) \qquad (2.08)$$

Under the conditions

(1) $u = 0, \quad \omega_y = \omega_o \quad$ at $\quad z = 0$
(2) $u \rightarrow U_o \quad$ as $\quad z \rightarrow \infty$

Eqs. (2.07) and (2.08) become

$$\frac{\omega_y}{\omega_o} = e^{-\mathbb{R}\,z/l_o} \qquad (2.09)$$

$$\frac{u}{U_o} = [1 - e^{-\mathbb{R}\,z/l_o}] \qquad (2.10)$$

in which

$$\mathbb{R} = \frac{k_2 U_o l_o}{\nu} \qquad (2.11)$$

Eq. (2.09) demonstrates clearly that while ω_y varies with z close to the surface, as z/l_o gets large for a given $\mathbb{R}$ or as $\mathbb{R}$ gets large for a given z/l_o, ω_y approaches zero. We thus see that the Euler equations are valid beyond a certain distance, δ/l_o, from a solid boundary. The distance, δ, is called the boundary layer thickness and is usually defined by

$$z = \delta \quad \text{when} \quad \frac{u}{U_o} = 0.99$$

This boundary layer, containing essentially all of the viscous effects, decreases in thickness as the Reynolds number increases and vice-versa.

We may redefine the Reynolds number of interest in these problems as:

$$\mathbb{R}_z = \frac{k_2 U_o z}{\nu} \qquad (2.12)$$

which has the special value, $\mathbb{R}_\delta$, at $z = \delta$. The velocity distribution of Eq. (2.10) can then be written

$$\frac{u}{U_o} = [1 - e^{-\mathbb{R}_z}] \qquad (2.13)$$

which is sketched in Fig. 2.2.

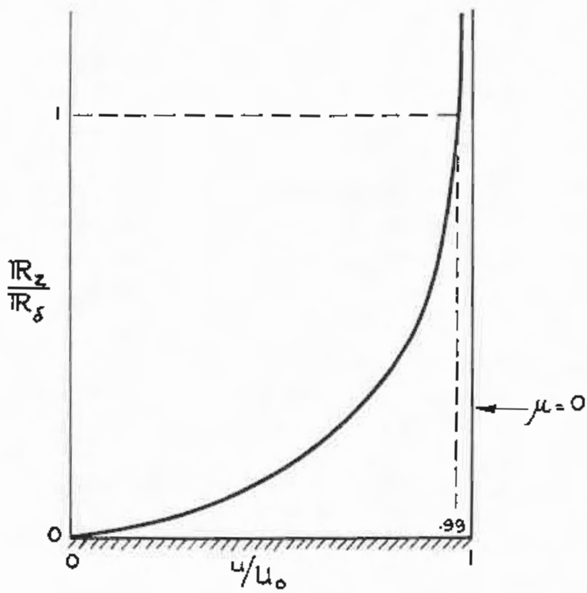

FIG. 2.2 VELOCITY DISTRIBUTION NEAR SOLID BOUNDARY

Recognition of the fact that the Reynolds number, $\mathbb{R}_z$, represents the ratio of inertia to viscous forces allows us to identify three regimes of flow:

(1) *Large* $\mathbb{R}_z$—Inertia forces predominate throughout the flow field. This is satisfied in a real fluid only where the vorticity is constant and thus cannot apply near a vorticity-generating surface. Eqs. (2.04),

the Euler equations, govern this regime. If the vorticity is zero this type of flow is known as irrotational flow.

(2) *Very low* $\mathbb{R}_z$—Viscous forces predominate throughout the flow field. Vorticity transport occurs principally by diffusion and Eq. (2.05) may be approximated by:

$$\nabla^2 \omega_y = 0 \tag{2.14}$$

This is called creeping flow.

(3) *Intermediate* $\mathbb{R}_z$—Viscous and inertia forces are both important and the flow obeys the complete Navier-Stokes equations. This type of flow is called boundary layer flow.

Utilizing Fig. 2.2, this categorization can be summarized:

Location	*Type of Flow*	*Applicable Dynamic Equations*
(z = 0 at shearing surface)		
$\mathbb{R}_z > \mathbb{R}_\delta$	Irrotational Flow	Euler Equations (2.04)
$\mathbb{R}_z \ll \mathbb{R}_\delta$	Creeping Flow	$\nabla^2 \omega = 0$
Elsewhere	Boundary Layer Flow	Complete Navier-Stokes Eq. (1.52)

Irrotational Flow

Let us look first at the case of very large Reynolds numbers where

$$\nabla^2 u = \nabla^2 v = \nabla^2 w = 0$$

In two-dimensions, the applicable equations of motion are thus Eq. (2.04). If we take the special but most common case of constant vorticity, that of irrotational motion, we have from the definition of ω_y:

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \tag{2.15}$$

It can be shown (1) that Eq. (2.15) is a necessary and sufficient condition for the existence of a scalar function of space and time, Φ, called the velocity potential which satisfies the relationships:

$$\nabla^2 \Phi = 0, \tag{2.16}$$

and

$$\left.\begin{aligned}\frac{\partial\Phi}{\partial x} &= -u \\ \frac{\partial\Phi}{\partial z} &= -w\end{aligned}\right\} \tag{2.17}$$

Considering only gravitational body forces we can use Eqs. (1.54), (2.15) and (2.17) to rewrite the Euler Eqs. (2.04):

$$\left.\begin{aligned}\frac{\partial}{\partial x}\left[-\frac{\partial\Phi}{\partial t}+\frac{1}{2}(u^2+w^2)+\frac{p}{\rho}\right]&=0 \\ \frac{\partial}{\partial z}\left[-\frac{\partial\Phi}{\partial t}+\frac{1}{2}(u^2+w^2)+\frac{p}{\rho}+gz\right]&=0\end{aligned}\right\} \tag{2.18}$$

in which form they are readily integrated to yield

$$\left.\begin{aligned}-\frac{\partial\Phi}{\partial t}+\frac{1}{2}(u^2+w^2)+\frac{p}{\rho}&=F_1(z,t) \\ -\frac{\partial\Phi}{\partial t}+\frac{1}{2}(u^2+w^2)+\frac{p}{\rho}+gz&=F_2(x,t)\end{aligned}\right\} \tag{2.19}$$

Subtracting equations (2.19) gives

$$gz = F_2(x,t) - F_1(z,t)$$

Since g is not a function of x, it is apparent that F_2 is a function of time alone and thus $F_1 = F_2(t) - gz$. Eqs. (2.19) are then reduced to the single relationship

$$-\frac{\partial\Phi}{\partial t}+\frac{1}{2}(u^2+w^2)+\frac{p}{\rho}+gz=F_2(t) \tag{2.20}$$

For steady flows Eq. (2.20) reduces to

$$\frac{1}{2}(u^2+w^2)+\frac{p}{\rho}+gz=\text{constant} \tag{2.20a}$$

which is the usual form of the steady state Bernoulli equation.

Equations (2.16) and (2.20) provide the means for solving the irrotational flow problem. In the general case, Eq. (2.16) is solved first for the only Φ which also satisfies the applicable boundary conditions. With Φ and thus the velocity components known, Eq. (2.20) may be solved for the pressure intensity, p, in terms of the unknown time function, $F_2(t)$. Since fluid motion is affected only by pressure gradients and $F_2(t)$ is a constant throughout the fluid at any time, t, the choice of $F_2(t)$ is evidently arbitrary. We may thus say

$$F_2(t) = 0 \tag{2.21}$$

without loss of generality.

This technique permits solution of many practical flow problems of interest to the civil engineer such as the mechanics of water waves, and pressure distribution on submerged objects.

Creeping Flow

When the viscous forces far outweigh the inertial forces (very low $\mathbb{R}_e$) we have seen that the Navier-Stokes equations reduce to a form in which the vorticity is a potential function, i.e.

$$\nabla^2 \omega = 0$$

We can also write Eqs. (1.52) for this case:

$$\left.\begin{aligned} \frac{\partial p}{\partial x} &= \mu \nabla^2 u \\ \frac{\partial p}{\partial y} &= \mu \nabla^2 v \\ \frac{\partial p}{\partial z} + \gamma &= \mu \nabla^2 w \end{aligned}\right\} \tag{2.22}$$

Differentiating the first of these by x, the second by y and the third by z we see that Eqs. (2.22) may be added to obtain

$$\nabla^2 p = \mu \nabla^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \tag{2.23}$$

but from continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

thus, in creeping flow the pressure is also a potential function, i.e.

$$\nabla^2 p = 0 \tag{2.24}$$

The theory of this flow classification is applicable to the solution of such civil engineering problems as flow in porous media and the settling of very fine sediments.

Boundary Layer Flow

It is indeed unfortunate that in the vast majority of engineering problems both inertia and viscous forces are important. One reason for this dismay has already been pointed out—no general solution to the very complex complete Navier-Stokes equations has been found. However, many practical problems involve conditions which permit simplification of the equations. Let us look first at an example of the type of simplification which will allow exact solutions.

Parallel Flow—

If two of the three velocity components are everywhere zero, the flow is called "parallel." If these two are v and w, Eqs. (1.34) and (1.52) become

$$\frac{\partial u}{\partial x} = 0 \tag{2.25}$$

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{2.26}$$

$$\left.\begin{aligned} \frac{\partial p}{\partial y} &= 0 \\ \frac{\partial p}{\partial z} &= -\gamma \end{aligned}\right\} \tag{2.27}$$

If the flow is steady and occurs between flat parallel walls a distance 2 b apart, Eq. (2.26) may be further reduced to

$$\frac{dp}{dx} = \mu \frac{d^2 u}{dy^2} \tag{2.28}$$

with boundary conditions $u = 0$ when $y = \pm b$. Since $\frac{\partial p}{\partial y} = 0$, $\frac{dp}{dx}$ is not a function of y and Eq. (2.28) can be integrated to give the velocity distribution

$$\left.\begin{aligned} u &= -\frac{1}{2\mu} \frac{dp}{dx} (b^2 - y^2) \\ \text{or} \quad \frac{u}{U_0} &= \left(1 - \frac{y^2}{b^2}\right) \end{aligned}\right\} \tag{2.29}$$

In this solution the Navier-Stokes equations were solved exactly since the boundary conditions permitted linearization without approximation. It should be noted that the velocity distribution obtained is parabolic rather than exponential as was given by the crude approximations leading to Eq. (2.13). Other practical examples of this type of exact solution can be found. More often however, the simplification which permits solution must involve an approximation. A classical and extremely important example of this technique is that originated by Prandtl for the study of boundary layers on a flat plate.

Boundary Layer Growth on a Flat Plate—

Considering an enclosed two-dimensional flow, the dimensionless Navier-Stokes equations (1.58) and the continuity equation (1.59) become:

$$\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial l_x} + W\frac{\partial U}{\partial l_z} = -\frac{\partial P_d}{\partial l_x} + \frac{1}{\mathbb{R}}\left(\frac{\partial^2 U}{\partial l_x^2} + \frac{\partial^2 U}{\partial l_z^2}\right) \tag{2.30}$$

$$1 \qquad 1\cdot 1 \qquad \delta\cdot\frac{1}{\delta} \qquad\qquad \delta^2 \qquad 1 \qquad \frac{1}{\delta^2}$$

$$\frac{\partial W}{\partial T} + U\frac{\partial W}{\partial l_x} + W\frac{\partial W}{\partial l_z} = -\frac{\partial P_d}{\partial l_z} + \frac{1}{\mathbb{R}}\left(\frac{\partial^2 W}{\partial l_x^2} + \frac{\partial^2 W}{\partial l_z^2}\right) \tag{2.31}$$

$$\delta \qquad 1\cdot\delta \qquad \delta\cdot 1 \qquad\qquad \delta^2 \qquad \delta \qquad \frac{1}{\delta}$$

$$\frac{\partial U}{\partial l_x} + \frac{\partial W}{\partial l_z} = 0 \tag{2.32}$$

$$1 \qquad 1$$

The no slip boundary condition requires

$$\begin{array}{ll} U = W = 0 & \text{at} \quad l_z = 0 \\ U = 1 & \text{at} \quad l_z \rightarrow \infty \end{array}$$

We will select the reference length, l_o, of Eqs. (1.57) such that $\partial U/\partial l_x$ has a magnitude of order 1. From Eq. (2.32) $\partial W/\partial l_z$ must then also be of order 1 and since W must go from 0 to its maximum value across the vertical distance, $l_z = \delta$, W must be of order δ. Assuming the local accelerations to be of the same order of magnitude as the convective accelerations we can then assign the orders of magnitude shown under the equations (2.30), (2.31) and (2.32). We note that $\frac{\partial P_d}{\partial l_z}$ is of order δ which means that the pressure has essentially its static distribution perpendicular to the boundary. Furthermore the pressure at $z = \delta$ may be determined from irrotational flow, thus it is regarded as a known function of l_x and T which is impressed upon the boundary layer by the external flow.

In the light of these arguments, Eqs. (2.30)-(2.32) can be approximated by the relations (returning to dimensional form):

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2} \tag{2.33}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{2.34}$$

with the boundary conditions

$$\begin{aligned} u = w = 0 \qquad & \text{at} \quad z = 0 \\ u = U_0(x,t) \qquad & \text{at} \quad z \to \infty \end{aligned}$$

The pressure intensity, p(x,t), in Eq. (2.33) may be determined from solution of the irrotational relation (i.e. Euler Eq.), approximately applicable beyond $z = \delta$:

$$\rho \frac{DU_0}{Dt} = -\frac{\partial p}{\partial x} \tag{2.35}$$

With Eqs. (2.33), (2.34) and (2.35) the variation in velocity from that given at some initial section can be determined as a result of the impressed potential flow.

This very important technique forms the basis for all analytical evaluations of viscous drag.

Even when legitimate approximations cannot be found which permit an analytic solution of a problem there may be a way out. As long as a problem can be formulated, it can (in principle) be solved for a specific set of conditions using numerical techniques and high-speed digital computation. It is the problem formulation which presents the real difficulty. We must remember that the Navier-Stokes equations were based upon an assumed stress-strain relationship modeled after that for an elastic solid but employing rate of strain instead of strain and substituting the molecular viscosity, μ, for the shear modulus, G. As Reynolds number increases, this so-called laminar relationship ceases rather abruptly to provide a sufficient definition of the stress-rate of strain phenomenon.

Stability of Laminar Flows

A physical system is said to be in a stable state when an arbitrary perturbation to the state is damped with time. It is unstable when the perturbation grows continuously, thereby changing the state of the system. Stability of a flow system such as is described by the Navier-Stokes equations depends upon the relative rate at which the energy of a disturbance is dissipated by viscosity and the rate at which energy is transferred to the disturbance from the mean flow.

If the velocities and pressures of a two-dimensional parallel flow are assumed given by

$$\left.\begin{aligned} u &= U + u' \\ w &= w' \\ p &= P + p' \end{aligned}\right\} \qquad (2.36)$$

in which the primed components represent perturbations from the time average values U and P, then the Navier-Stokes equations become:

$$\left.\begin{aligned} &\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + w'\frac{dU}{dz} + \frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{1}{\rho}\frac{\partial p'}{\partial x} \\ &\qquad\qquad = \nu\left[\frac{d^2U}{dz^2} + \nabla^2 u'\right] \\ &\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{1}{\rho}\frac{\partial p'}{\partial z} = \nu\nabla^2 w' \end{aligned}\right\} \qquad (2.37)$$

The only way energy can get from the mean flow to the perturbation is if mathematical coupling of the two motions exists in one or more terms of these equations. If no such coupling exists, then the two motions are independent. This coupling can be seen to exist in the non-linear inertia terms, $U\,\partial u'/\partial x$, $w'\,dU/dz$ and $U\,\partial w'/\partial x$. Damping of the perturbations can occur only through the viscous terms $\nu\nabla^2 u'$ and $\nu\nabla^2 w'$. It is thus expected that as the Reynolds number of the flow increases (and thus the ratio of inertia to viscous forces increases) a stable flow may become unstable. Furthermore, it should be noted that the inertial reaction varies directly with the temporal and spatial gradient of the perturbations, thus the stability should be sensitive to their frequency as well as to the Reynolds number of the mean flow.

Mathematical methods exist for the analysis of hydrodynamic stability and have been applied successfully (see Schlichting (2)) to the prediction of the stability relationship of the flow in a laminar boundary layer. The result of this analysis which has been confirmed experimentally, is indicated qualitatively in Fig. 2.3.

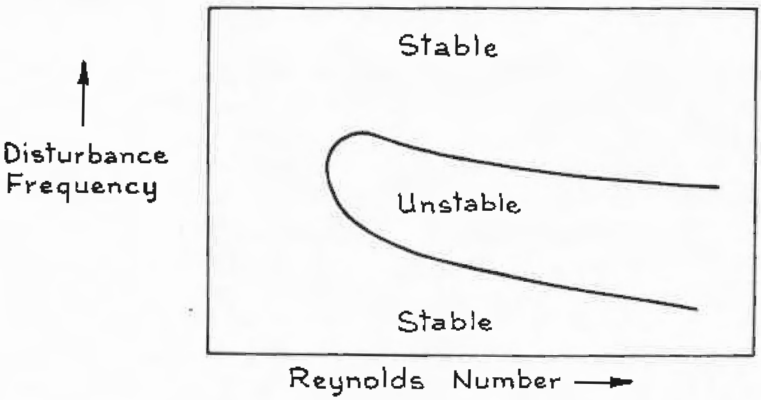

FIG. 2.3 STABILITY OF FLOW IN A LAMINAR BOUNDARY LAYER

Turbulence

In most practical situations, the flow is subjected continuously to a whole spectrum of disturbance frequencies arising from structural vibrations and geometrical irregularities. We have just seen that when the flow is in shear, these disturbances may be amplified. As Fig. 2.3 shows us, there is a Reynolds number below which none of these disturbances will be amplified. This value is so low, however, as to make the existence of laminar flows in civil engineering situations the rare exception rather than the rule.

The unstable disturbances grow in magnitude until something occurs to increase their rate of viscous dissipation sufficiently to create a new equilibrium state called a turbulent flow. This added dissipation occurs due to the molecular viscosity of the fluid, just as does the direct dissipation of the mean motion, however, the strain rates involved depend upon the kinematic structure of the disturbances rather than on that of the mean flow alone. The equations of motion governing the turbulent state of flow are therefore evidently different from those of Navier and Stokes which govern the laminar state of Newtonian fluids. It should be noted that the turbulent velocity fluctuations are generally smaller than the mean velocity, nevertheless they have a very important effect upon such fundamental characteristics as energy loss, drag and mixing.

Because of the probabilistic nature of the turbulent perturbations it is convenient to adopt an analytical model for their behavior which can take advantage of statistical methods of analysis. As we have already done, all dependent variables, f, (i.e. velocity, pressure, force, piezometric head, etc.) will be written in terms of a time average or mean quantity $\bar{f}$ and a random fluctuating component (or, deviation), f′, such that

$$f = \bar{f} + f' \tag{2.38}$$

where

$$\overline{f'} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f'\,dt = 0 \tag{2.39}$$

A convenient statistical measure of the deviation, f′, is given by the root-mean-square (RMS):

$$\text{RMS } f' = \sqrt{\overline{(f')^2}} = \left[\lim_{T\to\infty}\frac{1}{T}\int_0^T (f')^2\,dt\right]^{1/2} \tag{2.40}$$

In operating with the equations that are to follow it will be helpful to remember the following rules for taking the time average:

$$\left.\begin{aligned}
\overline{(\bar{f}+f')} &= \bar{f} + \overline{f'} = \bar{f}\\
\overline{(\bar{f}+f')(\bar{g}+g')} &= \bar{f}\cdot\bar{g} + \overline{f'\cdot\bar{g}} + \overline{g'\cdot\bar{f}} + \overline{f'\cdot g'} = \bar{f}\cdot\bar{g} + \overline{f'\cdot g'}\\
\overline{\frac{\partial(\bar{f}+f')}{\partial s}} &= \overline{\frac{\partial\bar{f}}{\partial s}} + \overline{\frac{\partial f'}{\partial s}} = \frac{\partial\bar{f}}{\partial s} + \frac{\partial\overline{f'}}{\partial s} = \frac{\partial\bar{f}}{\partial s}
\end{aligned}\right\} \tag{2.41}$$

Reynolds Equations

Since conversion of mechanical flow energy into heat (i.e. dissipation) can ultimately occur only through the action of the molecular viscosity, all motions, if examined in fine enough detail, must actually be laminar and hence must satisfy the Navier-Stokes equations. However, to make use of this fact the Navier-Stokes equations must be written in terms of the microstructure (turbulence) of the flow and not just in terms of the mean quantities. Let us substitute the quantities

$$\left.\begin{aligned} u &= \bar{u} + u' \\ v &= \bar{v} + v' \\ w &= \bar{w} + w' \\ p &= \bar{p} + p' \end{aligned}\right\} \tag{2.42}$$

into the Navier-Stokes (1.52) and continuity (1.34) equations. We will carry this out in detail for the x component of motion in order to generate some familiarity with the reasoning involved:

Using Eqs. (2.42), the first of Eqs. (1.52) can be expanded to give:

$$\left.\begin{aligned} \frac{\partial \bar{u}}{\partial t} + \frac{\partial u'}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial u'}{\partial x} + u'\frac{\partial \bar{u}}{\partial x} + u'\frac{\partial u'}{\partial x} & \\ + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{v}\frac{\partial u'}{\partial y} + v'\frac{\partial \bar{u}}{\partial y} + v'\frac{\partial u'}{\partial y} & \\ + \bar{w}\frac{\partial \bar{u}}{\partial z} + \bar{w}\frac{\partial u'}{\partial z} + w'\frac{\partial \bar{u}}{\partial z} + w'\frac{\partial u'}{\partial z} & \\ = \frac{\bar{X}}{\rho} + \frac{X'}{\rho} - \frac{\partial \bar{p}}{\rho \partial x} - \frac{\partial p'}{\rho \partial x} & \\ + \nu \nabla^2 \bar{u} + \nu \nabla^2 u' & \end{aligned}\right\} \tag{2.43}$$

Let us now consider only those flows in which the mean values do not vary with time. Then, if we take the time average of Eq. (2.43) using the rules of Eq. (2.41), we obtain:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} + \overline{u'\frac{\partial u'}{\partial x}} + \overline{v'\frac{\partial u'}{\partial y}} + \overline{w'\frac{\partial u'}{\partial z}} = \frac{\bar{X}}{\rho} - \frac{\partial \bar{p}}{\rho \partial x} + \nu \nabla^2 \bar{u} \tag{2.44}$$

Subtracting Eq. (2.44) from Eq. (2.43) gives (again for "steady" flows) a similar equation in terms of the fluctuating components:

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + u'\frac{\partial \bar{u}}{\partial x} + u'\frac{\partial u'}{\partial x} - \overline{u'\frac{\partial u'}{\partial x}}$$

$$+ \bar{v}\frac{\partial u'}{\partial y} + v'\frac{\partial \bar{u}}{\partial y} + v'\frac{\partial u'}{\partial y} - \overline{v'\frac{\partial u'}{\partial y}}$$

$$+ \bar{w}\frac{\partial u'}{\partial z} + w'\frac{\partial \bar{u}}{\partial z} + w'\frac{\partial u'}{\partial z} - \overline{w'\frac{\partial u'}{\partial z}}$$

$$= \frac{X'}{\rho} - \frac{\partial p'}{\rho \partial x} + \nu \nabla^2 u' \qquad (2.45)$$

Performing similar operations on the continuity equation we obtain:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \qquad (2.46)$$

and

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \qquad (2.47)$$

Turbulent flows can often be handled for engineering purposes by understanding only the mean flow in detail (although real advances can come only through an understanding of the fluctuations). For this reason we will restrict our attention to Eq. (2.44) and the reader is referred to the work of Rouse (3) for discussion of the equations governing the fluctuating components. Multiplying Eq. (2.47) by the fluctuating component, u', and averaging we get

$$\overline{u'\frac{\partial u'}{\partial x}} + \overline{u'\frac{\partial v'}{\partial y}} + \overline{u'\frac{\partial w'}{\partial z}} = 0 \qquad (2.48)$$

which may be added (since it is equal to zero) to Eq. (2.44) to give for the mean flow:

$$\rho\frac{D\bar{u}}{Dt} + \rho\left[\overline{u'\frac{\partial u'}{\partial x}} + \overline{u'\frac{\partial u'}{\partial x}}\right] + \rho\left[\overline{u'\frac{\partial v'}{\partial y}} + \overline{v'\frac{\partial u'}{\partial y}}\right]$$

$$+ \rho\left[\overline{w'\frac{\partial u'}{\partial z}} + \overline{u'\frac{\partial w'}{\partial z}}\right] = \bar{X} - \frac{\partial \bar{p}}{\partial x} + \mu \nabla^2 \bar{u} \qquad (2.49)$$

Note for example that:

$$\overline{u'\frac{\partial v'}{\partial y}}+\overline{v'\frac{\partial u'}{\partial y}}=\overline{u'\frac{\partial v'}{\partial y}+v'\frac{\partial u'}{\partial y}}=\overline{\frac{\partial(u'v')}{\partial y}}=\frac{\partial\overline{u'v'}}{\partial y}$$

We can then move the turbulent convective accelerations to the right hand side and write Eq. (2.49) in the revealing form:

$$\left.\begin{aligned}\rho\left(\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}+\bar{w}\frac{\partial\bar{u}}{\partial z}\right)=\bar{X}-\frac{\partial\bar{p}}{\partial x}&\\+\frac{\partial}{\partial x}\left(\mu\frac{\partial\bar{u}}{\partial x}-\rho\,\overline{u'u'}\right)&\\+\frac{\partial}{\partial y}\left(\mu\frac{\partial\bar{u}}{\partial y}-\rho\,\overline{u'v'}\right)&\\+\frac{\partial}{\partial z}\left(\mu\frac{\partial\bar{u}}{\partial z}-\rho\,\overline{u'w'}\right)&\end{aligned}\right\}\tag{2.50}$$

When written in the form of Eq. (2.50), the fluctuating terms in the mean flow equation of motion can be interpreted as stresses. These stresses are known as apparent or virtual stresses of turbulent flow or more commonly perhaps as Reynolds stresses. They may be very large when compared with their companion terms for the mean viscous shear, but they are still inertia forces and cannot account for energy dissipation since ultimately this must involve viscous action.

Remembering that we are considering only flows which are steady in the mean we can write the three Reynolds equations

$$\left.\begin{aligned}\rho\frac{D\bar{u}}{Dt}&=\bar{X}-\frac{\partial\bar{p}}{\partial x}+\mu\nabla^2\bar{u}-\rho\left[\frac{\partial\overline{u'^2}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}\right]\\\rho\frac{D\bar{v}}{Dt}&=Y-\frac{\partial\bar{p}}{\partial y}+\mu\nabla^2\bar{v}-\rho\left[\frac{\partial\overline{u'v'}}{\partial x}+\frac{\partial\overline{v'^2}}{\partial y}+\frac{\partial\overline{v'w'}}{\partial z}\right]\\\rho\frac{D\bar{w}}{Dt}&=\bar{Z}-\frac{\partial\bar{p}}{\partial z}+\mu\nabla^2\bar{v}-\rho\left[\frac{\partial\overline{u'w'}}{\partial x}+\frac{\partial\overline{v'w'}}{\partial y}+\frac{\partial\overline{w'^2}}{\partial z}\right]\end{aligned}\right\}\tag{2.51}$$

These equations, together with the continuity equations (2.46) and (2.47), form the tools for the exact solution of a turbulent flow problem. Unfortunately however, three new dependent variables, u′, v′ and w′, have been added with the addition of only one extra equation, (2.47). Therefore, the mean flow cannot be determined, for a given set of boundary conditions, without some relationship between the mean and fluctuating components of velocity. The search for such relationships has formed the basis for the bulk of analytical and experimental research on turbulent flows.

Classification of Turbulent Flows

Consider a turbulent flow near a solid boundary. As a consequence of the no slip condition there can be no turbulent fluctuation in the plane of the boundary at the boundary. Furthermore, the fluctuating component normal to the boundary must vanish unless the latter is permeable. Turbulence is thus zero at a solid boundary and will be small in its immediate vicinity. It follows that at and very near a solid boundary the viscous stresses predominate over the Reynolds stresses and the flow is laminar. The zone in which this occurs is called the laminar sub-layer. As distance from the wall increases, the effect of the wall in suppressing turbulence decreases and the Reynolds stresses

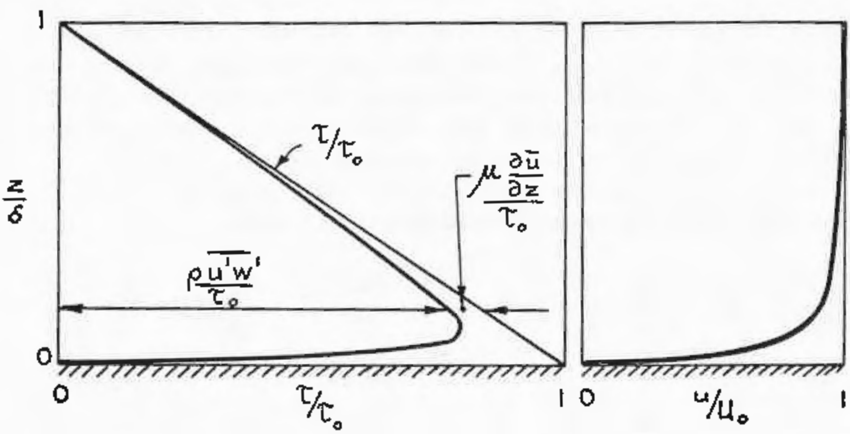

FIG. 2.4 VELOCITY AND SHEAR DISTRIBUTION IN TURBULENT FLOW NEAR A WALL

grow in magnitude. This is indicated schematically in Fig. 2.4. At the same time the viscous stresses are decreasing due to the reduction in velocity gradient with distance from the wall. Eventually the Reynolds stresses predominate and this zone of the boundary layer is fully turbulent.

A turbulent flow such as just described is classified as wall turbulence due to action of the boundary in suppressing the fluctuations. It is further classified as a turbulent shear flow due to the non-zero value of the Reynolds stresses.

Turbulence in the early wake of an object would also be a turbulent shear flow since the turbulence originates at a shearing surface as in the case of the wall boundary layer. In the case of the wake, however, the shear surface is a fluid surface, the separation streamline, at which the fluctuations need not vanish. Such flows are called free turbulence due to the absence of boundary effects.

Flows in which the turbulence has the same structure at all points are called homogeneous. Two-dimensional flows are homogeneous in at least one direction.

A special class of homogeneous flows is that in which the turbulence structure does not vary with direction at any point. These are called isotropic. It is the nature of free turbulent fields which are decaying under the action of viscosity to approach the condition of isotropy.

If a flow is truly isotropic, the fluctuation components must be statistically independent. The mean value of the product of two statistically independent random variables is the product of their mean values or zero. Thus, for isotropic turbulence

$$\left.\begin{aligned} & u'^2 = v'^2 = w'^2 \\ \text{and} \quad & \overline{u'v'} = \overline{u'w'} = \overline{v'w'} = 0 \end{aligned}\right\} \tag{2.51a}$$

and the fluctuating terms vanish from the Reynolds equations. In actuality however, even though the correlations, $\overline{u'v'}$, etc. may be zero, the flow decays in the direction of the mean velocity, thus the gradient of the $\overline{u'^2}$ term is finite.

Turbulent Transport Processes

To fully understand the importance of turbulence in every aspect of human life, some attention must be paid to the manner in which energy, momentum and mass are transferred from one spot to another in a fluid medium. The mechanisms for this transfer are radiation, conduction and convection.

Radiation, a wavelike phenomenon, is of interest in civil engineering problems which involve the compressibility of the fluid or the propagation of gravity waves. The modes of major interest are conduction and convection.

Conduction—The conduction process is molecular in scale. The molecules of any solid, liquid or gaseous substance are in continual random motion, the intensity of which is determined by the temperature of the given substance. Because of this random molecular motion there will be an interchange of molecules between layers. This interchange will be effective, in the average, over a length which is known as the molecular mean free path. These conduction processes are sometimes termed molecular diffusion.

If adjacent layers are at different temperatures, this exchange will produce a net transport of heat; if they are in motion at different mean velocities, this will produce a net transverse transport of momentum. If we are dealing with impure (multicomponent) fluid systems, different concentrations of the impurity in adjacent layers will produce a net transport of mass (called ordinary diffusion).

Convection—The convection process is macroscopic in comparison with the microscopic nature of conduction. Once again the transport depends upon a gradient. That is, if a static body of fluid is subjected to a gradient of piezometric head, flow will take place which convects (i.e. carries with it) energy, mass and momentum. For purposes of this discussion we will restrict ourselves to turbulent convection or turbulent diffusion which is that transport which occurs due to the turbulent fluctuations. It is conceptually convenient (although not strictly correct physically) to look upon turbulent motion as a system of eddies (vortexes) of varying scale (size) and intensity (rotational velocity) superimposed on the mean flow. Under such a model macroscopic packets of fluid are moved about at random thereby convecting their cargoes of energy, mass and momentum from spot to spot at a rate which is, for the same mean flow conditions, many times greater than that which occurs due to the molecular conduction processes just discussed.

A similarity is observed in the above descriptions of molecular and turbulent diffusion which provides the key to a phenomenological model relating the mean flow and the Reynolds stresses.

Let us call the amount of the property (per unit mass of fluid) whose transfer is to be studied, s, and let us assume (only for convenience) that in our given flow this quantity varies in only the z direction. This is indicated in Fig. 2.5. Because of molecular motion

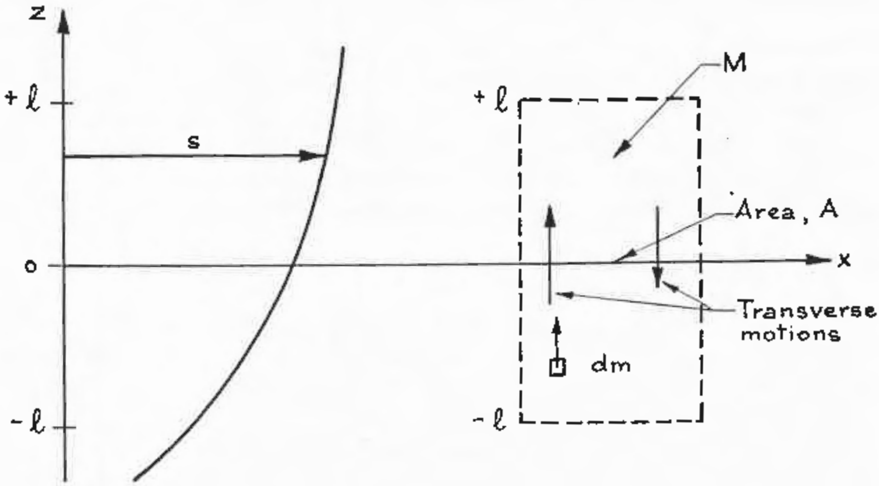

FIG. 2.5 DEFINITION SKETCH FOR CONDUCTION AND TURBULENT CONVECTION

and also turbulence (if present) a vertical exchange of fluid packets is taking place through any horizontal area, A, at elevation, z. The distance traveled by these fluid packets before taking on the character of their new surroundings is assumed to be a random quantity, the mean value of which we will call l. The amount of s carried by a parcel of mass, dm, and passing through A will depend upon the point of origin of the motion of the parcel. This is because of the assumed gradient of s in the direction of the random motion. Statistically speaking no particle lying outside the limits $+l > z > -l$ will pass through A. The average time needed for the parcel to travel the distance l will be called t.

The net upward transport, S, of s per unit of time and area may then be written

$$S = \frac{M/2}{Atl}\left[\int_{-l}^{0} s(z)\,dz - \int_{0}^{+l} s(z)\,dz\right]$$

Expanding s(z) in a Taylor series and integrating

$$S = -\frac{\rho l^2}{t}\left(\frac{\partial s}{\partial z}\right)_{z=0} = -K\left(\frac{\partial s}{\partial z}\right)_{z=0} \qquad (2.52)$$

When speaking of molecular processes the quantity $\frac{l^2}{t}$ is a property of the medium and in general varies with both temperature and pressure. If the process is heat conduction then $\partial s/\partial z$ becomes $\partial T/\partial z$ and K is the thermal conductivity. If the process is molecular momentum transfer, s becomes the momentum per unit mass (i.e. the velocity) and S, the momentum flux per unit of area. The constant K is then the molecular viscosity, $\nu = \mu/\rho$. For $s = \overline{u}$, Eq. (2.52) becomes:

$$S = \tau_{zx} = -\mu\frac{\partial \overline{u}}{\partial z}$$

The minus sign signifies that the net transfer of momentum takes place "down" the velocity gradient thereby applying a force in the positive x direction to fluid below surface A.

Eq. (2.52) could be applied to the case of turbulent convection by imagining the length, l, to be a mixing length. In this case the time required for the particle to travel the distance, l, would be

$$t = \frac{l}{v'}$$

and the change in x component of mean velocity experienced by an excursion in the positive z direction would be

$$u' = -l\frac{\partial \overline{u}}{\partial z}.$$

Using these two relations Eq. (2.52) becomes, in the average

$$\tau_{zx} = +\rho\,\overline{u'v'}$$

which is the Reynolds stress. It is clear now that K is not a property of the fluid for the case of Reynolds stresses but must reflect the local kinematics of the flow. We know that for $\overline{u'v'} \neq 0$, u' and v' must be related. Assuming this relationship to be a simple proportion we include the proportionality factor in the definition of l to write:

$$\tau = \rho l^2 \left|\frac{\partial \overline{u}}{\partial z}\right| \frac{\partial \overline{u}}{\partial z} = \rho\varepsilon\frac{\partial \overline{u}}{\partial z} \qquad (2.53)$$

The quantity ε is often called the kinematic eddy viscosity. From arguments of dynamic similarity, Kármán introduced the assumption that

$$l = \varkappa \frac{\partial \bar{u}/\partial z}{\partial^2 \bar{u}/\partial z^2} \tag{2.54}$$

in which the coefficient, $\varkappa$, has come to be known as the Kármán Universal Constant. Combining (2.53) and (2.54) we can write

$$\sqrt{\frac{\tau}{\rho}} = -\varkappa \frac{(\partial \bar{u}/\partial z)^2}{\partial^2 \bar{u}/\partial z^2} \tag{2.55}$$

which can be integrated once the function $\tau(z)$ is known.

To summarize this discussion of turbulent transport processes:

1. Diffusion, either molecular or turbulent, requires a gradient of the diffusant and takes place in the direction of decreasing concentration of the diffusant.
2. For a given gradient in a given medium the "depth of penetration" of the diffusion process will be directly proportional to the diffusivity, K.
3. The rate of transfer under a given gradient in a given medium is directly proportional to the diffusivity, K.
4. Experiments have shown the eddy diffusivity (eddy viscosity) to be of the order 10^3 to 10^7 times larger than the molecular diffusivity (molecular viscosity).
5. The total transfer under any circumstances is equal to the sum of the transfers by molecular and turbulent means.

It is also interesting to note that experiments indicate a difference in the turbulent transfer coefficients for mass and momentum under the same flow conditions.

Turbulent Energy Spectrum

The energy in a random process such as the turbulent velocity fluctuations is distributed over a continuous spectrum of frequencies, f, or alternatively wave numbers, k, where

$$f(\text{cps}) \sim k \sim \frac{2\pi}{L}$$

It has been established experimentally that the wave number of the turbulence generated by the mean flow in the process of doing work against the Reynolds stresses is of the order of the boundary scale. For example, as an irrotational flow of a real fluid passes through a screen, a free turbulent shear flow is created by the multitude of wakes which has a predominant wave number determined by the bar size of the screen. Imagine this initial disturbance to be a simple harmonic motion. We can imagine this sinusoidal disturbance as the input to a black box the output of which is the turbulent velocity fluctuation at some point further downstream of the screen. If the black box performs a linear operation on the input, the output will be monochromatic also. We know, however, that the equation of motion is highly non-linear, thus it is reasonable to expect this spectral transfer function to be non-linear also. In this case a monochromatic input is distorted so that the output must contain the fundamental (input) frequency in addition to other, higher, harmonics. Thus, as these original eddies move downstream, they continually give up energy to the generation of eddies at slightly

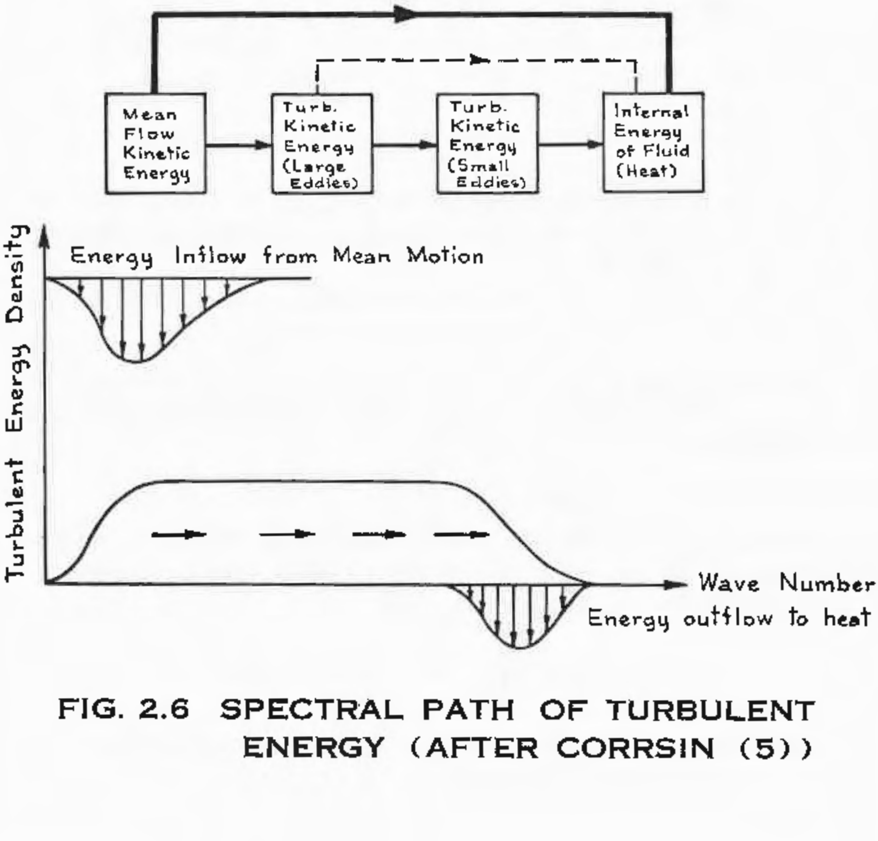

FIG. 2.6 SPECTRAL PATH OF TURBULENT ENERGY (AFTER CORRSIN (5))

higher wave number. These eddies then generate still smaller eddies and so on. Eventually the scale of the eddies becomes so small (thus decreasing the turbulence Reynolds number, ε/ν) that the viscous forces predominate and the kinetic energy of the fluctuations is converted ("dissipated") into heat. This energy path is indicated schematically in Fig. 2.6 which is taken from the superb discussion by Corrsin (5).

Turbulent Shear Flows

Wall Turbulence—

Turbulence due to the influence of solid boundaries is probably of primary interest to civil engineers. Under the assumption that τ is constant with z and is equal to the boundary value, τ_o, Eq. (2.55) can be integrated to give the familiar logarithmic distribution of mean velocity:

$$\frac{\bar{u}}{\sqrt{\tau_o/\rho}} = \frac{1}{\varkappa} \ln z + C$$

Experiments in Newtonian fluids indicate $\varkappa$ to be about 0.4 and the constant C to vary with the geometry of the channel. It is curious that the relationship holds well in parallel flows where τ varies linearly with z.

Free Turbulence—

Free turbulent shear flows are commonly found in wakes or jets where a separation stream line exists. This separation streamline is initially a line of abrupt discontinuity in mean velocity. The high shear at this discontinuity generates intense turbulence which then diffuses laterally. In the case of wall turbulence an equilibrium mean velocity profile was eventually reached because the turbulence-generating surface was continuous in the direction of the mean motion. In the case of free turbulence, however, the lateral transport of momentum acts to soften the discontinuity in mean velocity which is serving as the turbulence generator. The diffusion process thus causes, in the direction of the mean flow, a continuous modification of the mean velocity distribution and a continuous widening of the wake or jet. This is illustrated by the sketch of Fig. 2.7.

Analytical solutions proceed as in the case of boundary layer analysis. The general procedure is as follows:

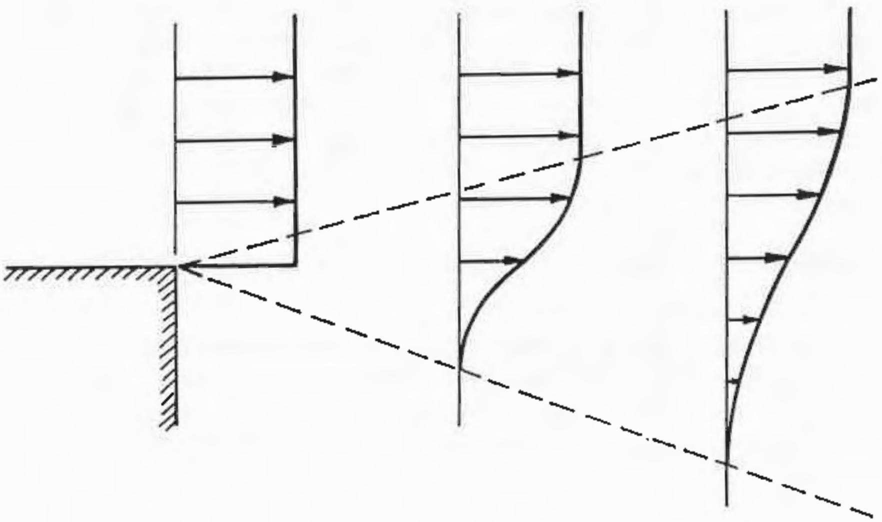

FIG. 2.7 MEAN VELOCITY GRADIENTS IN FREE TURBULENT SHEAR FLOW

1. Neglect longitudinal velocity variations in comparison with lateral.

2. Assume rate of wake spreading is very gradual. (With 1 and 2 the pressure can be assumed constant everywhere.)

3. Neglect viscous stresses in comparison with Reynolds stresses. For a two-dimensional flow we then have (Eqs. (2.51) and 2.46)):

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -\frac{\partial (\overline{u'w'})}{\partial z}$$

and

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{w}}{\partial z} = 0$$

4. Assume similar distribution of mean velocity at all x.

5. Assume the lateral distribution of some characteristic of the mean flow or of the turbulence.

6. Assume some relation between mean flow and turbulence.

If two diffusing shear layers meet, such as will happen at some point downstream in a jet of finite diameter or behind a bluff body of finite thickness, the diffusion process is altered and a separate analytical solution must be sought. The early stage of such diffusion is called the zone of flow establishment and holds up to the intersection of the two processes. Downstream of this lies the zone of established flow.

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